Nonequilibrium thermal entanglement for simple qubit systems

I. Sinayskiy, N. Pumulo, and F. Petruccione

Quantum Research Group, School of Physics and National Institute for Theoretical Physics, University of KwaZulu-Natal, Durban, 4001, South Africa,
e-mails: sinayskiy@ukzn.ac.za; nate95@gmail.com; petruccione@ukzn.ac.za

Recibido el 24 de enero de 2011; aceptado el 24 de marzo de 2011

The dynamics of simple qubit systems in a chain configuration coupled at both ends to separate bosonic baths at different temperatures is studied. An exact analytical solution of the master equation in the Born-Markov approximation for the reduced density matrix of the qubit system is constructed. The unique non-equilibrium stationary state for the long time behavior of the reduced density matrix in obtained. Dynamical and steady state properties of the concurrence between the first and the last spin are studied.

Keywords: Open quantum systems; decoherence; entanglement.

Se estudia la dinámica de sistemas simples de qubit en una configuración en cadena acoplada en ambos extremos con baños bosónicos a diferentes temperaturas. Se construye una solución analítica exacta para la ecuación maestra en la aproximación de Born-Markov para la matriz densidad reducida del sistema de qubit. Se obtiene el único estado de no equilibrio estacionario para el comportamiento a tiempos largos de la matriz densidad. Se estudian las propiedades dinámicas y estacionarias de la concurrencia entre el primer y el último espín.

Descriptores: Sistemas cuánticos abiertos; decoherencia; entrelazamiento.

PACS: 03.67.Mn; 03.65.Yz; 65.40G-

1. Introduction

In describing real physical systems one should always take into account the influence of the surroundings. The study of open systems is particularly important for understanding processes in quantum physics [1]. Usually, the unavoidable interaction with the environment destroys quantum correlations in the system. However, some times dissipation is not only for creation of entanglement in the system [2–5], but also for construction of non-trivial complex states of the system [6, 7]. An entanglement created by the interaction with thermal reservoir is called thermal entanglement. During the interaction with the thermal environment at temperature \( T \) the system is evolving towards its steady state given by the Gibbs distribution \( \rho \rightarrow e^{-\hat{H}_S/k_B T}/\text{Tr}[e^{-\hat{H}_S/k_B T}] \) and sometimes this state is entangled. In this paper another kind of thermal entanglement will be discussed. Here, the system will be driven to the steady state by the interaction with two reservoirs at different temperatures [2, 3, 8, 9].

The paper is organized as follows. In Sec. 2 the model of a qubit system in a chain configuration coupled at both ends to bosonic baths at different temperatures is introduced. The general strategy of derivation of the master equation for the reduced density matrix in the Born-Markov limit is presented. In Sec. 3 the two qubit case is considered. The master equation and its steady state solution are constructed. In Sec. 4 the master equation for a three qubit case is obtained. The steady state solution in this case is constructed. Finally, in Section V results and conclusions are presented.

2. Model

We consider a qubit system in a chain configuration. The first and the last qubits of the chain are coupled to separate bosonic baths at different temperatures. In this paper units are chosen in a such way that \( \hbar = k_B = 1 \). The total Hamiltonian of the system is given by

\[
\hat{H} = \hat{H}_S + \hat{H}_{B1} + \hat{H}_{BN} + \hat{H}_{SB1} + \hat{H}_{SBN},
\]

where \( \hat{H}_S \) is the Hamiltonian of the qubit subsystem,

\[
\hat{H}_S = \sum_{i=1}^{N} \frac{\epsilon_i}{2} \hat{\sigma}_i^z + K \sum_{i=1}^{N-1} (\hat{\sigma}_i^+ \hat{\sigma}_{i+1}^- + \hat{\sigma}_i^- \hat{\sigma}_{i+1}^+).
\]

The thermal reservoirs are described by an infinite set of harmonic oscillators, the Hamiltonians of the reservoirs coupled to the first qubit \((j = 1)\) and to the last qubit \((j = N)\) are given by

\[
\hat{H}_{Bj} = \sum_n \omega_{n,j} \hat{b}_{n,j}^\dagger \hat{b}_{n,j}.
\]

The interaction between the qubit subsystem and the bosonic baths is described by the Jaynes Cummings type Hamiltonian in the rotating wave approximation

\[
\hat{H}_{SBj} = \hat{\sigma}_j^+ \sum_n g^{(j)}_n \hat{b}_{n,j} + \hat{\sigma}_j^- \sum_n g^{(j)*}_n \hat{b}_{n,j}^\dagger,
\]

of course, \( \hat{\sigma}_j^\pm, \hat{\sigma}_j^z \) are the well-known Pauli matrices and \( \hat{b}_{n,j}^\dagger, \hat{b}_{n,j} \) denote bosonic creation and annihilation operators.

The dynamics of the total system (qubits plus reservoirs) is described by the Liouville equation

\[
\frac{d}{dt} \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)].
\]

After performing the Born-Markov approximation [1] the equation for the reduced density matrix (qubit subsystem...
only) in the interaction picture takes the following form:

\[
\frac{d}{dt}\hat{\rho}_S(t) = -\int_0^\infty ds \text{Tr}_B \left[ \hat{H}_S^I(t), \hat{H}_S^I(t-s), \hat{\rho}_S(t) \otimes \hat{\rho}_B(0) \right],
\]

where, the operator \( \hat{\rho}_B(0) \) denotes the initial state of the reservoirs

\[
\hat{\rho}_B(0) = \frac{e^{-\beta_B H_{B1}}}{\text{Tr}[e^{-\beta_B H_{B1}}]} \otimes \frac{e^{-\beta_B H_{BN}}}{\text{Tr}[e^{-\beta_B H_{BN}}]},
\]

and \( \hat{H}_S^{I} = \hat{H}_S^{I} \). After performing the rotating wave approximation over the rapidly oscillating term in the master equation one gets:

\[
\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{H}_S, \hat{\rho}_S(t)] + \sum_\omega \sum_\alpha,\beta \sum_{i,j} \gamma^{i,j}_{\alpha,\beta}(\omega) \left( \hat{\gamma}_j^I(\omega) \hat{\rho}_S(t) \hat{\gamma}_j^{I\dagger}(\omega) - \frac{1}{2} \left[ \hat{\gamma}_j^{I\dagger}(\omega) \hat{\gamma}_j^I(\omega), \hat{\rho}_S(t) \right]_+ \right). \quad (8)
\]

To obtain the master Eq. (8) it assumed that the system-environment interaction has the form

\[
\hat{H}_{SB} = \sum_\alpha \hat{\gamma}_j^I \otimes \hat{f}_j^I + \text{h.c.}, \quad (9)
\]

the operators \( \hat{\gamma}_j^I \) and \( \hat{f}_j^I \) acts on the qubit system and the reservoir degrees of freedom, respectively. In the master Eq. (8) a Lamb-type renormalization Hamiltonian was neglected and decay rates \( \gamma^{i,j}_{\alpha,\beta}(\omega) \) are given by the Fourier image of the bath correlation functions:

\[
\gamma^{i,j}_{\alpha,\beta}(\omega) = \int dse^{i\omega s} \langle \hat{f}_j^I(s) \hat{f}_j^{I\dagger}(0) \rangle. \quad (10)
\]

On should note that, in Eq. (8) \( \sum_\omega \) is a sum over all Bohr frequencies of the system. In Eqs. (8) and (10) indexes \( \alpha \) and \( \beta \) refer to decomposition of the interaction Hamiltonian into eigenoperators of the system Hamiltonian \( \hat{H}_S \) and indexes \( i \) and \( j \) label the baths. The transition operators \( \hat{\gamma}_j^I(\omega) \) originate from the decomposition of the operator \( \hat{V}_j^I \) in the basis of the eigenoperators of the system Hamiltonian \( \hat{H}_S \). If one denotes the eigenvalues of the Hamiltonian \( \hat{H}_S \) by \( \epsilon \) and corresponding projection operator as \( \Pi(\epsilon) \) then:

\[
\hat{V}_j^I(\omega) = \sum_{\epsilon = \omega} \epsilon \hat{\Pi}(\epsilon) \hat{V}_j^I \hat{\Pi}(\epsilon). \quad (11)
\]

In other words each operator \( \hat{V}_j^I(\omega) \) satisfies the following equation:

\[
[\hat{H}_S, \hat{V}_j^I(\omega)] = -\omega \hat{V}_j^I(\omega), \quad (12)
\]

where \( \omega \) corresponding frequency of transition. For the sake of simplicity and exact solvability in the paper we will considered only two cases: two and three qubit systems and symmetric Hamiltonian of the qubit system (2), i.e., for any \( i \) all \( \epsilon_i = \epsilon \).

### 3. Two qubit system

In the two qubit case the Hamiltonian of the qubit subsystem (2) takes a simple form

\[
\hat{H}_S = \frac{\epsilon}{2} (\hat{\sigma}_1^+ + \hat{\sigma}_2^+) + K (\hat{\sigma}_1^- \hat{\sigma}_2^- + \hat{\sigma}_1^+ \hat{\sigma}_2^+) \quad (13)
\]

and can easily be diagonalized with eigenvectors and corresponding eigenvalues given below

\[
|m_1\rangle = |00\rangle, m_1 = -\frac{\epsilon}{2}, \quad (14)
\]
\[
|m_2\rangle = |11\rangle, m_2 = \frac{\epsilon}{2}, \quad (15)
\]
\[
|m_3\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle), m_3 = K, \quad (16)
\]
\[
|m_4\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), m_4 = -K. \quad (17)
\]

In this case the master equation (8) takes the following form

\[
\frac{d\hat{\rho}}{dt} = -i[\hat{H}_S, \hat{\rho}] + \mathcal{L}_1(\hat{\rho}) + \mathcal{L}_2(\hat{\rho}), \quad (18)
\]

where the superoperators \( \mathcal{L}_i \) describe dissipation to the corresponding reservoir:

\[
\mathcal{L}_i(\hat{\rho}) = \sum_{j=1}^2 \gamma^{(i)}(\omega_j) \left( \hat{\gamma}_j^I(\omega) \hat{\rho} \hat{\gamma}_j^{I\dagger}(\omega) - \frac{1}{2} \left[ \hat{\gamma}_j^{I\dagger}(\omega) \hat{\gamma}_j^I(\omega), \hat{\rho} \right]_+ \right) + \gamma^{(i)}(\omega_j) \left( \hat{\gamma}_j^{I\dagger}(\omega) \hat{\rho} \hat{\gamma}_j^I(\omega) - \frac{1}{2} \left[ \hat{\gamma}_j^I(\omega) \hat{\gamma}_j^{I\dagger}(\omega), \hat{\rho} \right]_+ \right). \quad (19)
\]

The transition operators \( \hat{V}_j^I(\omega) \) are

\[
\hat{V}_1^I(\omega) = \frac{1}{\sqrt{2}} (|m_1\rangle\langle m_3| + |m_4\rangle\langle m_2|), \quad (20)
\]
\[
\hat{V}_2^I(\omega) = \frac{1}{\sqrt{2}} (|m_4\rangle\langle m_2| - |m_1\rangle\langle m_4|), \quad (21)
\]
\[
\hat{V}_1^2(\omega) = \frac{1}{\sqrt{2}} (|m_4\rangle\langle m_3| - |m_1\rangle\langle m_2|), \quad (22)
\]
\[
\hat{V}_2^2(\omega) = \frac{1}{\sqrt{2}} (|m_3\rangle\langle m_2| + |m_1\rangle\langle m_4|), \quad (23)
\]

and the corresponding frequencies of transition are

\[
\omega_1 = \omega_2 = \omega_1 = \epsilon - K, \quad (24)
\]

\[ \omega_1 = \omega_2 = \omega_2 = \omega = \epsilon + K. \] (25)

In this paper reservoirs are assumed to be represented by the infinite set of harmonic oscillators so that the decay rates read
\[ \gamma_{ij}^{(i)}(\omega) \equiv \gamma^{(i)}(\omega) \delta_{ij} = \gamma_i n_i(\omega) \delta_{ij} = \frac{\gamma_i}{e^{\beta_i \omega} - 1} \delta_{ij}, \] (26)
\[ \gamma^{(i)}(\omega) = \gamma_i n_i(\omega) e^{\beta_i \omega}, \] (27)
where \( \gamma_i \) is the relaxation rate given by the spectral density of the reservoir.

The master Eq. (18) is solved exactly and details of the solution (including non-symmetric case \( \epsilon_1 \neq \epsilon_2 \)) can be found in Ref. 2. In this paper we will need only the non-equilibrium stationary state of the two qubit system for the comparison with the three qubit case. After some straightforward algebra the steady state of the two qubit system in the standard basis \( |00\rangle, |01\rangle, |10\rangle, |11\rangle \) has the form:

\[ \rho_\infty = \frac{1}{X_1 X_2} \begin{pmatrix} X_1^{X_2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} (X_1^{X_2} + X_1^{X_2}) & \frac{1}{2} (X_1^{X_2} - X_1^{X_2}) & 0 \\ 0 & \frac{1}{2} (X_1^{X_2} - X_1^{X_2}) & \frac{1}{2} (X_1^{X_2} + X_1^{X_2}) & 0 \\ 0 & 0 & 0 & X_1^{X_2} \end{pmatrix}, \] (28)

and corresponding eigenvalues \( m_i \):
\[ m_1 = -m_4 = -\frac{3\epsilon}{2}, \] (38)
\[ m_2 = -m_3 = -\frac{\epsilon}{2}, \] (39)
\[ m_5 = -m_8 = -\frac{\epsilon}{2} - \sqrt{2}K, \] (40)
\[ m_6 = -m_7 = \frac{\epsilon}{2} - \sqrt{2}K. \] (41)

4. Three qubit system

The three qubit case is also exactly solvable. Following the same strategy one can find eigenvalues and eigenvectors for the Hamiltonian \( H_S \) (2) in the case of three qubits:
\[ |m_1\rangle = |000\rangle, \] (30)
\[ |m_2\rangle = \frac{|001\rangle - |100\rangle}{\sqrt{2}}, \] (31)
\[ |m_3\rangle = \frac{|011\rangle - |110\rangle}{\sqrt{2}}, \] (32)
\[ |m_4\rangle = |111\rangle, \] (33)
\[ |m_5\rangle = \frac{|100\rangle - \sqrt{2} |010\rangle + |001\rangle}{2}, \] (34)
\[ |m_6\rangle = \frac{|110\rangle - \sqrt{2} |101\rangle + |011\rangle}{2}, \] (35)
\[ |m_7\rangle = \frac{|100\rangle + \sqrt{2} |010\rangle + |001\rangle}{2}, \] (36)
\[ |m_8\rangle = \frac{|110\rangle + \sqrt{2} |101\rangle + |011\rangle}{2}. \] (37)

In this case the master equation (8) for the reduced density matrix reads
\[ \frac{d\hat{\rho}}{dt} = -i[\hat{H}_S, \hat{\rho}] + L_1(\hat{\rho}) + L_3(\hat{\rho}) \] (42)
where the superoperators \( L_i \) are given by
\[ L_i(\hat{\rho}) = \sum_{j=1}^{3} \gamma^{(i)}(-\omega_j) \left( \hat{V}_j^{(i)}(\omega) \hat{\rho} \hat{V}^{i\dagger}_j(\omega) \right. \]
\[ - \frac{1}{2} [\hat{V}^{i\dagger}_j(\omega) \hat{V}_j^{i\dagger}(\omega), \hat{\rho} + ] \]
\[ + \gamma^{(i)}(\omega_j) \left( \hat{V}_j^{i\dagger}(\omega) \hat{\rho} \hat{V}^{i\dagger}_j(\omega) \right) \]
\[ - \frac{1}{2} [\hat{V}^{i\dagger}_j(\omega) \hat{V}_j^{i\dagger}(\omega), \hat{\rho} + ] \]. (43)

There are three transition operators \( \hat{V}_j^i(\omega) \) for each reservoir, namely,
\[ \hat{V}_1^1(\omega) = \frac{1}{\sqrt{2}} (-|m_1\rangle\langle m_2| + |m_3\rangle\langle m_4| - |m_5\rangle\langle m_6| + |m_7\rangle\langle m_8|), \] (44)
\[ \hat{V}_2^1(\omega) = \frac{1}{2} (|m_1\rangle\langle m_5| - |m_2\rangle\langle m_6| - |m_7\rangle\langle m_8| + |m_8\rangle\langle m_4|), \] (45)
The non-equilibrium steady-state of the two-qubit system is presented. Figure 2 addresses the dependence of the steady-state concurrence in the two-qubit and three-qubit cases on the parameters of the Hamiltonians. The master Eq. (42) is solved exactly. The details of the solution can be found in Ref. 9. One can study the dynamics of the concurrence of the system for a certain initial state of the system. In Fig. 1 the dynamics of the concurrence between first and third qubit for initial \( |W_3\rangle \) state is presented.

\[
|W_3\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle).
\]

For the all three cases presented in Fig. 1 one can identify two time scales of the dynamics, the first timescale is the time of dissipation of the entanglement created by the \( XX \)-interaction between qubits and the second timescale is the creation of the thermal entanglement. The phenomena of disappearing and reappearing of the entanglement is called sudden death and sudden birth of entanglement [11].

Using the exact solution of the master Eq. (42) one can find the long-time behavior of the density matrix. It is possible to show that in the basis of eigenvectors \( \{|m_i\}\) of the qubit Hamiltonian \( H_S \) the non-equilibrium steady-state of the reduced density matrix of the qubit system will be diagonal

\[
\rho_{\infty}^{ij} = \frac{1}{X_1 X_2 X_3} \begin{pmatrix}
X_1^+ X_2^+ X_3^+ \\
X_1^+ X_2^+ X_3^- \\
X_1^+ X_2^- X_3^+ \\
X_1^+ X_2^- X_3^- \\
X_1^- X_2^+ X_3^+ \\
X_1^- X_2^+ X_3^- \\
X_1^- X_2^- X_3^+ \\
X_1^- X_2^- X_3^- 
\end{pmatrix},
\]

where \( X_i^\pm = e^{i\omega_i} \pm e^{i\omega_i} \) and \( X_i = X_i^+ + X_i^- \).

Using this non-equilibrium steady-state one can analyze the concurrence in the system. In Figs. 2 and 3 a comparison of the steady state concurrence in the two qubit and three qubit case is presented. Figure 2 addresses the dependence of the steady state concurrence form the reservoir temperatures in the equilibrium case. Figure 3 presents the dependence of the concurrence on the Hamiltonian parameters.

\[
\begin{align*}
\hat{V}_3^1 (\omega) &= \frac{1}{2} \left( |m_1\rangle \langle m_7| + |m_2\rangle \langle m_8| + |m_5\rangle \langle m_3| + |m_6\rangle \langle m_4| \right), \\
\hat{V}_3^2 (\omega) &= \frac{1}{\sqrt{2}} \left( |m_1\rangle \langle m_2| - |m_3\rangle \langle m_4| - |m_5\rangle \langle m_6| + |m_7\rangle \langle m_8| \right), \\
\hat{V}_3^3 (\omega) &= \frac{1}{2} \left( |m_1\rangle \langle m_5| + |m_2\rangle \langle m_6| + |m_7\rangle \langle m_3| + |m_8\rangle \langle m_4| \right), \\
\hat{V}_3^4 (\omega) &= \frac{1}{2} \left( |m_1\rangle \langle m_7| - |m_2\rangle \langle m_8| - |m_5\rangle \langle m_3| + |m_6\rangle \langle m_4| \right),
\end{align*}
\]

Figure 1. Dynamics of the concurrence between the first and the third qubit for different temperatures of the reservoirs. The initial state of the three qubit system is the \( |W_3\rangle \) state. Curve (a) corresponds to \( \beta_1 = \beta_3 = 10 \), curve (b) to \( \beta_1 = \beta_3 = 5 \), curve (c) to \( \beta_1 = 5 \), \( \beta_3 = 3 \); all the other parameters are the same: \( \epsilon = 3/2, \kappa = 1, \gamma_1 = \gamma_N = 1/50 \). For a convenience curves (b) and (c) are shifted up on the Y axes by 1 and 2 units, respectively.

Figure 2. Comparison of the steady-state entanglement for two qubit and three qubit systems in the thermal equilibrium. Curves (1a), (2a) and (3a) correspond to the two qubit system and curves (1b), (2b) and (3b) to the two qubit system. Curves (1a) and (1b) correspond to \( \epsilon/K = \sqrt{2} \), curves (2a) and (2b) correspond to \( \epsilon/K = 3/2 \) and curves (3a) and (3b) correspond to \( \epsilon/K = 2 \). For a convenience curves 2 and 3 are shifted up on the Y axes by 0.3 and 0.5 units, respectively.
For the reservoirs of low temperature (curves $a$ in Fig. 1 one can see the system approaching equilibrium. It is important to note that the non-equilibrium steady state of a two qubit system is higher than in the two qubit case. But as in the symmetric two qubit case the steady state concurrence reaches its maximal value in the equilibrium case.

The steady state concurrence in the three qubit case is higher than in a two qubit one. In two qubit one.

In Fig. 3 we show a comparison between steady state concurrence for two and three qubit system in the non-equilibrium case. One can see that in all cases there are intervals of the temperatures of the reservoirs for which the steady state concurrence in the three qubit case is higher than in a two qubit one. But as in the symmetric two qubit case the steady state entanglement reaches its maximal value in the equilibrium case.

In conclusion, we have presented an analytical expression for a two and three qubit system in a chain configuration coupled to bosonic baths at different temperatures. The dynamics of the system is studied and it is shown that the system convergences to a non-equilibrium steady state. The dynamics of entanglement is analyzed and a comparison of the steady state concurrence of two and three qubit systems is performed. It is found that there is a range of parameters in which the three qubit system contains more quantum correlations in the steady state than two qubit one.

This work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and National Research Foundation.