Unidirectional two-dimensional periodic waveguides

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In this work we describe a general method to construct two-dimensional periodically modulated waveguides with constant cross section. The constant cross section makes the motion of a point particle moving inside the waveguide to be unidirectional. We also analyze the classical dynamics of non-interacting particles moving inside our unidirectional waveguides by means of Poincaré maps.

Keywords: Waveguides; chaos; unidirectionality.

En este trabajo describimos un método general para construir guías de ondas bidimensionales moduladas periódicamente cuya sección transversal es constante. El que la sección transversal sea constante hace que el movimiento de partículas puntuales que se mueven en la guía de ondas sea unidireccional. También analizamos la dinámica clásica de partículas no interactuantes que se mueven dentro de las guías de ondas unidireccionales utilizando mapeos de Poincaré.

Descriptores: Guías de ondas; caos; unidireccionalidad.

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1. Introduction

Billiard tables provide a prominent example of Hamiltonian systems with very rich dynamical properties [1, 2]. The dynamics of billiards can range from integrable to completely chaotic and it is entirely defined by the shape of the billiard’s boundary. Examples of integrable billiards are rectangles and ellipses while the stadium and the Sinai billiard are the most popular billiards developing full chaos. Most billiards with smooth convex boundaries have a phase space consisting of KAM-islands merged into chaotic components, a situation known as mixed chaos [3]. However, recently, two types of billiards having well-defined disjoint phase space components were introduced: mushroom billiards [4] and unidirectional billiards [5–10] whose key characteristic is to have constant cross section. The phase space of mushroom billiards is split in one regular and one chaotic component. While in unidirectional billiards the phase space is divided in two invariant sets formed by orbits moving in opposite directions.

Up to now there are two prescriptions for unidirectional billiards: circular billiards subjected to special deformations [5, 6] and billiards formed by semi-circular and straight channel segments [7–10] (also known as track billiards). In particular, extended track billiards are proposed as a model for bent optical fibers since they form unidirectional waveguides [7, 8]. It is important to stress that extended track billiards are very easy to construct and analyze due to their integrable (semi-circular and straight) components but they lack of smooth boundaries. Non-smooth boundaries in billiards produces singularities in some classical dynamical quantities and introduce the undesired effect of diffraction in their wave counterparts.

To overcome the above problems, in this paper we introduce a class of generic unidirectional waveguides. That is, our unidirectional waveguides have smooth walls defined by arbitrary functions.

The organization of this paper is as follows. In the next Section we define the procedure to construct unidirectional waveguides having arbitrary profile shape. Then, in Sec. 3 we specialize the profile of the waveguide to a periodically modulated function that allows us to increase/decrease the complexity of the waveguide. In Sec. 4 we analyze, by the use of Poincaré maps, the dynamics of particles moving inside our periodically modulated unidirectional waveguides. Finally, we draw our conclusions in Sec. 5.

2. Construction of unidirectional waveguides

We consider a two-dimensional waveguide with constant cross section $d$, i.e. a unidirectional waveguide, defined by the functions $f(x)$ and $g(x)$. See Fig. 1. We assume that the function $f(x)$ is known, therefore the problem reduces to find the function $g(x)$.

Let $\vec{c}(t) = (x(t), f(t))$ be the parametrization of $f(x)$, then the normal vector at any point on $\vec{c}(t)$ is given by:

$$\vec{N}(t) = \left( x''(t)f'^2(t) - x'(t)f'(t)f''(t), f''(t)x'^2(t) - f'(t)x'(t)x''(t) \right).$$

(1)

Here, the prime and double prime indicate the first and second derivative, respectively, with respect to the parameter $t$. We implicitly assume that $f(x)$ is of class $C^2$, at least, and that $f''(t) \neq 0$. Then, the parametrization of the function $g(x)$ is given by

$$\vec{c}(t) = \vec{c}(t) + c\vec{N}(t),$$

(2)

where $\vec{N}(t) = \vec{N}(t)/||\vec{N}(t)||$ and $c$ can be positive or negative. So, given any function $f(x)$, by the use of Eq. (2), it...
we have observed [11] that $M$ drives the function $f(x)$ from smooth ($M \sim 1$) to rough-like ($M \sim 100$). This form for $f(x)$ will allow us to construct periodically modulated unidirectional waveguides with some degree of complexity driven by $M$.

Once the function $f(x)$ is specified by Eq. (3), we can apply Eq. (2) to construct $g(x)$: We choose the parametrization $x(t) = t$, so that $f(t) = f(x)$ and

$$
\overrightarrow{N}(t) = \left( \sum_{m=1}^{M} A_m m \sin(mt) \sum_{m=1}^{M} A_m m^2 \cos(mt) \right),
$$

which leads to the general expression for the parametrization of $g(x)$:

$$
\overrightarrow{c}(t) = \left( t + d \frac{A_m \cos(mt)}{\sqrt{\left[ \sum_{m=1}^{M} A_m \sin(mt) \right]^2 + 1}} \right).
$$

In the following Section we give examples of periodic unidirectional waveguides constructed from Eq. (5).

3.1. Examples of periodic unidirectional waveguides

3.1.1. Case $M = 1$

We first consider the simplest case where the function $f(x)$ is given by a single cosine function, i.e. $M = 1$ in Eq. (3). Then, the parametrization of $g(x)$ according to Eq. (5) is given by

$$
\overrightarrow{c}(t) = \left( t + d \frac{A \sin(t)}{\sqrt{A^2 \sin^2(t) + 1}} \right),
$$

$$
A \cos(t) + \frac{d}{\sqrt{A^2 \sin^2(t) + 1}}.
$$

In Fig. 2 we plot the unidirectional waveguide resulting from $f(x) = A \cos(x)$ with $d = 1$ [Fig. 2(a)] and $d = -1.75$ [Fig. 2(b)]; in both cases $A = 1$. Notice that while in Fig. 2(a) the function $g(x)$ is smooth, in Fig. 2(b) $g(x)$ is a multivalued function. That means that not all values of $d$ are allowed in the construction of unidirectional waveguides. If we write $\overrightarrow{c}(t) = (x(t), g(t))$, for $g(t)$ to be a smooth function we need

$$
\frac{dx(t)}{dt} \geq 0.
$$
This condition applied to the cross section reads as
\[ d \leq \left[ A^2 \sin^2(t) + 1 \right]^{3/2} [A \cos(t)]^{-1}. \] (8)

Then, by evaluating the left part of condition (8) in the interval \(-\pi < t \leq \pi\) and considering the possibility of negative values of \(d\), we get the following bounds for \(d\): \(-1/A \leq d \leq 1/A\). That is, \(|d|\) should be smaller or equal to \(d_{\text{max}} = 1/A\) for \(g(x)\) to be a univalued function.

3.1.2. Case \(M > 1\)

For any \(M > 1\) condition (7) applied to the cross section leads to
\[ |d| \leq \frac{\left( \sum_{m=1}^{M} A_m m \sin(mt) \right)^2 + 1}{\sum_{m=1}^{M} A_m m^2 \cos(mt)} = d_{\text{max}}, \] (9)

which has to be evaluated numerically for every set of amplitudes \(A_m\). In Fig. 3 we plot \(\langle d_{\text{max}} \rangle\) as a function of \(M\). The average is taken over 100 realizations of \(f(x)\). We used 100 different random sequences of amplitudes \(A_m\) to calculate \(\langle d_{\text{max}} \rangle\). It is clear from Fig. 3 that \(\langle d_{\text{max}} \rangle\) vs. \(M\) follows a power-law behavior given by \(3.2 M^{-1.6}\). Then, we can rewrite condition (9) as
\[ |d| \lesssim \langle d_{\text{max}} \rangle \approx \frac{3.2}{M^{1.6}}. \] (10)

Notice that the larger the value of \(M\) (i.e. the more complex the geometry of the waveguide) the smaller the value of \(d_{\text{max}}\).

As examples, in Fig. 4 we plot unidirectional waveguides for \(M = 2, 5,\) and 10. In all cases we used \(d = d_{\text{max}}\).

4. Dynamics

Now, we consider non-interacting point particles moving inside the periodic unidirectional waveguide defined in the previous Section. The particles change their direction of motion when colliding with the waveguide walls. We assume that such collisions produce specular reflections on the particle’s trajectories.

In Fig. 5 we show the trajectories of 10 particles moving inside a waveguide with \(f(x) = A \cos(x)\). The particles are launched from the left side of the waveguide with the same initial angle, \(\theta_i = 0\), and slightly different initial positions.

Note that up to the first 6 collisions, the trajectories follow a beam-like pattern; i.e. the particles follow similar (in shape and length) paths. But after that the trajectories become complicated and the particles follow different paths. Moreover, the particles always move to the right: the signature of unidirectionality.

Even though a picture like Fig. 5 is very illustrative, to obtain an overview of the dynamics of particles moving inside the waveguide we construct Poincaré maps. The Poincaré map is a phase portrait obtained from the orbits of a representative set of initial conditions as they cross, in a specified direction, a given surface of section in phase space [3]. Here
we choose \( f(x) \), the lower wall of the waveguide, as the surface of section. Thus, each time a particle impinges on that wall we plot \( s \), the position along \( f(x) \), and the sinus of the angle \( \theta \); where \( \theta \) is the angle the trajectory makes with the normal to \( f(x) \) at \( x \).

In Fig. 6 we show a typical Poincaré map for the periodic unidirectional waveguide with \( f(x) = A \cos(x) \) (case \( M = 1 \)). This map was generated by the trajectories of several particles all with initial angles \( \theta_i > 0 \). Note that the points in phase space lie on one half of the phase space only. That is, since the particles are launched from left to right \( (\theta_i > 0) \) and the waveguide is unidirectional, their trajectories are confined to the part of the phase space having \( \theta > 0 \). This illustrates the separation of left-to-right and right-to-left motion: the way unidirectionality manifests in phase space.

The Poincaré map of Fig. 6 corresponds to the waveguide of Fig. 5 which, in particular, generates full chaos. However, for a given set of parameters \((d, A)\) the most likely situation is to observe a mixed phase space (co-existence of chaotic and regular regions). See some examples in Fig. 7 where the Poincaré maps are generated by trajectories launched from left to right \( (\theta_i > 0) \) and from right to left \( (\theta_i < 0) \), so the complete phase space is explored.

We want to note that, in contrast to standard waveguides (see for example [12]), for unidirectional waveguides the islands in phase space do not correspond to bounded motion. Instead they act as transporting islands [13]. As an example, in Fig. 8 we show a set of trajectories corresponding to the

![Figure 6](image1.png)

**Figure 6.** Poincaré map for the periodic unidirectional waveguide with \( f(x) = A \cos(x) \). \( d = \pi/4 \) and \( A = 1/d \) were used. \( s \) is normalized to the length of one period of \( f(x) \).

![Figure 7](image2.png)

**Figure 7.** Poincaré maps for the periodic unidirectional waveguide with \( f(x) = A \cos(x) \). \( d = 2\pi, 4\pi, \) and \( 1.8\pi \) (from left to right) and \( A = 1/d \). \( s \) is normalized to the length of one period of \( f(x) \).

![Figure 8](image3.png)

**Figure 8.** Set of regular trajectories for the periodic unidirectional waveguide whose Poincaré map is in Fig. 7(right). The initial conditions for these trajectories were chosen to lie inside the regular islands with \( \theta > 0 \).

![Figure 9](image4.png)

**Figure 9.** Poincaré maps for periodic unidirectional waveguides with \( M = 2, 4, \) and \( 8 \) (from left to right). \( d = d_{\text{max}} \) in all cases: \( d_{\text{max}} = 0.523, 0.351, \) and \( 0.104 \) (from left to right). \( s \) is normalized to the length of one period of \( f(x) \).
islands of the Poincaré map in Fig. 7(right). Clearly, the corresponding particles travel along the waveguide in a ballistic-like manner.

Finally, in Fig. 9, we show Poincaré maps for periodic unidirectional waveguides with $M > 1$. For $M > 1$ we have observed that the larger the value of $M$ (i) the less likely the appearance of islands in phase space; and (ii) the larger the region of bouncing-ball-like motion (see the regular-like phase space regions with $\sin(\theta) < 0.25$ in the Poincaré maps of Fig. 9). Bouncing-ball-like motion also happens in waveguides with $M = 1$ but the corresponding region in phase space is so thin that it is hardly visible in the Poincaré maps of Fig. 7.

5. Conclusions

In this paper we have introduced unidirectional waveguides with smooth walls defined by arbitrary functions. In particular, we explored the case of periodically modulated waveguides and study the dynamics of point particles moving inside them by means of Poincaré maps. We found that by increasing the complexity of our unidirectional waveguides the bouncing-ball-like motion is highly privileged. This may result in the subdiffusion of particles, an effect we plan to verify soon.

Also, in a future stage of this project we plan to study the effects of unidirectionality in the quantum version of the periodically modulated waveguides introduced here.

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1. A is chosen such that $|f(x)| \leq 1$.
2. The variables $(s, \sin(\theta))$ are known as Birkhoff variables, where $\sin(\theta)$ is the dimensionless tangential velocity of the particle.
3. In fact, by the use of Chirikov’s overlapping resonance criteria applied to an approximate Poincaré map for this waveguide, full chaos is already expected for the set of parameters used in Fig. 6.