Multifractal analysis of time series from CA by means of the wavelet transform

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In this work we study the time series of some elementary cellular automata (ECA) for possible multifractal behavior by means of the multifractal detrended fluctuation analysis (MF-DFA) based on the discrete orthogonal wavelet transform. The literature gives a variety of methods to compute the singularity spectrum, but the interest to consider wavelet methods is that they are more stable. We illustrate our results for two representative ECA rules: 90 and 150, and for a time series that considers the backward evolution of the cellular automaton rule 90.

Keywords: Multifractal; detrended fluctuation analysis; cellular automata; wavelets.

1. Introduction

It is well known that many natural systems exhibit complex dynamics described by long-range power laws. At the present time, a number of different algorithms are well established to analyze the singular behavior that may be hidden in time series data, such as the wavelet transform modulus maxima method (WTMM) [1-5], the structure function method [2], the detrended fluctuation analysis (DFA) [6] and its variants [7-9]. DFA is a method used to analyze the behavior of the average fluctuations of the data at different scales after removing the local trends. In some cases the output of such systems, which corresponds to a fluctuating time series, may be characterized or quantified by a spectrum of exponents called the multifractal spectrum. The multifractal formalism (MF) to characterize processes, measures, and functions, introduced by Halsey and collaborators [10], opened a new direction for the search of dimension type characteristics of dynamical systems in the form of spectra for dimensions that reflect both the geometric and dynamical structure of non-linear systems. In 2002, Kantelhardt et al. [7] provided a generalization of DFA to the case of multifractal time series. Subsequently, the latter method started to be widely employed in the literature under the name of MF-DFA. Kantelhardt wrote a recent review of the techniques used in processing the fractal and MF time series [8].

On the other hand, as already mentioned, a lot of research has been done on fractal signals and objects with wavelet transforms (WTs) because the multiscale decompositions implied by the WTs are well adapted to evaluate typical self-similarity properties. The efficiency of WTs as ‘mathematical microscopes’ for capturing the local scaling properties of fractals have been noticed since more than two decades [11]. In fact, the interest in wavelet methods is that they are numerically more stable [4]. In 1993, Bacry et al. [3] developed this method based on the definition of a partition function in terms of the WTMM. They demonstrated that the singularity spectrum for a Bernoulli measure or a fractal distribution can be readily determined from the scaling behavior of such a partition function and similar results have been proved for more general measures by Murguía and Urias [12].

It is thus no wonder that there are current efforts towards merging the WTs with DFA procedures [13,14] as a natural union of powerful tools for quantifying the scaling properties of the fluctuations. In this paper, based on this unifying standpoint, which we call WMF-DFA, we focus on the MF properties of two representative time series of elementary cellular automata (ECA) with periodic boundary conditions, as well as the representative time series of an application based on a rule-90 cellular automaton. A previous work dedicated to the MF features of ECA [15] has analyzed the time series of random walk processes generated by some of the ECA evolution rules and not directly to the ECA time series as we do here. In addition, Nagler and Claussen [16] mention in the final part of their work the possibility of considering their spectral analysis for MF signals instead of monofractal ones. We recall that many important applications of ECA are in cryptography, biology, and chemistry, where MF properties are to be expected. For example, an interpretation of ECA rules 90 and 150 can be made in the context of catalytic
depends on the present states of its left and right neighbors. Evolution from the neighborhood configuration (first row) to Table is the lookup table of rule 90, where it is specified the transformations [19]. Within this framework, one can write (DWT), which is one of the different forms of the wavelet processes [16], also the rule 126 can be used as a conceptual model of biological cell growth [17]. In fact, the ECA rule 90 has been considered as an intrinsic generator of randomness [18].

The structure of this paper is as follows. The following Section presents a general overview of cellular automata. In Sec. 3 is described the discrete orthogonal wavelet transform, and it is described the scheme of the wavelet multifractal detrended fluctuation analysis. Section 4 illustrates the analysis of MF-DFA applied to time series of ECA. Finally, conclusions are presented in Sec. 5.

2. Cellular Automata

An elementary cellular automaton(ECA) can be considered as a discrete dynamical that evolve at discrete time steps. An ECA is a cellular automata consisting of a chain of $N$ lattice sites with each site is denoted by an index $i$. Associated with each state $i$ is a dynamical variable $x_i$, which take only $k$ discrete values. Most of the studies have been done with $k = 2$, where $x_i = 0$ or 1. Therefore there are $2^N$ different states for these automata. One can see that the time, space, and states of this system take only discrete values. The ECA considered evolves according to the local rule

$$x_{n+1}^{t+1} = [x_n^t + r x_n^t + x_{n+1}^t] \mod 2,$$

where $r = 0$ defines rule 90 and $r = 1$ rule 150, respectively. Table is the lookup table of rule 90, where it is specified the evolution from the neighborhood configuration (first row) to the next state (second row), that is, the next state of $i$–th cell depends on the present states of its left and right neighbors.

In fact, a rule is numbered by the unsigned decimal equivalent of the binary expression in the second row. When the same rule is applied to update cells of ECA, such ECA are called uniform ECA; otherwise the ECA are called non-uniform or hybrids. It is important to observe that the evolution rules of ECA are determined by two main factors, the rule and the initial conditions.

<table>
<thead>
<tr>
<th>Neighborhood</th>
<th>111</th>
<th>110</th>
<th>101</th>
<th>100</th>
<th>011</th>
<th>010</th>
<th>001</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rule result</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

where the coefficients of the expansion are given by

$$d_n^m = \int_{-\infty}^{\infty} x(t) \psi_n^m(t) dt.$$  \hspace{1cm} (3)

The orthonormal basis functions are all dilations and translations of a function referred as the analyzing wavelet $\psi(t)$, and they can be expressed in the form

$$\psi_n^m(t) = 2^{m/2} \psi(2^m t - n),$$  \hspace{1cm} (4)

with $m$ and $n$ denoting the dilation and translation indices, respectively.

Mallat (1999) developed a computationally efficient method to compute efficiently (2) and (3). This method considers the multiresolution analysis (MRA). The MRA approach provides a general method for constructing orthogonal wavelet basis and leads to the implementation of the fast wavelet transform (FWT) algorithm. This algorithm connects, in an elegant way, wavelets and filter banks. A multiresolution signal decomposition of a signal $X$ is based on successive decomposition into a series of approximations and details, which become increasingly coarse. At the beginning, the signal is split into two parts, an approximation and a detail part, that together yield the original. The subdivision is such that the approximation signal contains low frequencies, while the detail signal collects the remaining high frequencies. By repeated application of this subdivision rule on the approximation, details of increasingly coarse resolution are separated out, while the approximation itself grows coarser and coarser. Associated with the wavelet function $\psi(t)$ is a corresponding scaling function, $\varphi(t)$, and scaling coefficients, $a_n^m$ [19]. The scaling and wavelet coefficients at scale $m$ can be computed from the scaling coefficients at the next finer scale $m + 1$ using

$$a_n^m = \sum_l h[l-2n] a_l^{m+1},$$  \hspace{1cm} (5)

$$d_n^m = \sum_l g[l-2n] a_l^{m+1},$$  \hspace{1cm} (6)

where $h[n]$ and $g[n]$ are typically called lowpass and highpass filters in the associated analysis filter bank. Equations (5) and (6) represent the fast wavelet transform (FWT) for computing (3). In fact, signals $a_n^m$ and $d_n^m$ are the convolutions of $a_n^{m+1}$ with the filters $h[n]$ and $g[n]$ followed by a downsampling of factor 2 [19]. Figure 1 (left panel) shows the frequency decomposition performed by the filters $h[n]$ and $g[n]$. 

Conversely, a reconstruction of the original scaling coefficients $a^{m+1}_n$ can be made from

$$a^{m+1}_n = \sum_l (h[2l-n]a^m_l + g[2l-n]x^m_l),$$  \hspace{1cm} (7)

a combination of the scaling and wavelet coefficients at a coarse scale. Equation (7) represents the inverse of FWT for computing (2). This corresponds to the synthesis filter bank. This part can be viewed as the discrete convolutions between the upsampled signal $a^n_n$ and the filters $h[n]$ and $g[n]$, that is, following an “upsampling” of factor 2 calculate the convolutions between the upsampled signal and the filters $h[n]$ and $g[n]$. The number of levels depends on the length of the signal. A signal with $2^k$ values can be decomposed into $k+1$ levels. To initialize the FWT, it is considered a discrete time signal $X = \{x[1], x[2], \ldots, x[N]\}$ of length $N = 2^M$. The first application of (5) and (6), beginning with $a^{M+1}_n = x[n]$, define the first level of the FWT of $X$. The process goes on, always adopting the “$m+1$” scaling coefficients to calculate the “$m$” scaling and wavelet coefficients. Iterating (5) and (6) $M$ times, the transformed signal consists of $M$ sets of wavelet coefficients at scales $m = 1, \ldots, M$, and a signal set of scaling coefficients at scale $M$. There are exactly $2^{(k-m)}$ wavelet coefficients $d^m_n$ at each scale $m$, and $2^{(k-M)}$ scaling coefficients $a^M_n$. The maximum number of iterations $M_{\text{max}} = k$. A three-level decomposition process of the FWT is shown in Fig. 1(right panel).

In a broad sense, with this approach, the low-pass coefficients capture the trend and the high-pass coefficients keep track of the fluctuations in the data. The scaling and wavelet functions are naturally endowed with an appropriate window size, which manifests in the scale index or level, and hence can capture the local averages and differences, in a window of one’s choice.

As is discussed in Ref. 13, 19, and 20, some degree of regularity is useful on the wavelet basis for the representation to be well behaved. To achieve this, a wavelet function should have $n$ vanishing moments. A wavelet is said to have $n$ vanishing moments, which will be denoted as $\psi_n(x)$, if and only if it satisfies

$$\int_{-\infty}^{\infty} t^k \psi(t) dt = 0$$

for $k = 0, 1, \ldots, n - 1$ and

$$\int_{-\infty}^{\infty} t^n \psi(t) dt \neq 0$$

for $k = n$. This means that a wavelet with $n$ vanishing moments is orthogonal to all polynomials up to order $n - 1$. Thus, the DWT of $x(t)$ performed with a wavelet $\psi(t)$ with $n$ vanishing moments is nothing else but a “smoothed version” of the $n$–th derivative of $x(t)$ on various scales. This important property helps detrending the data. In fact, when someone is interested to measure the local regularity of a signal this concept is crucial [19,20].

### 3.1. WMF-DFA Algorithm

We are interested in revealing the MF properties [10] of ECA. To separate the trend from fluctuations in the ECA time series, we follow the discrete wavelet method proposed by Manimaran et al. [13]. This method exploits the fact that the low-pass version resembles the original data in an “averaged” manner in different resolutions. Instead of a polynomial fit, we consider the different versions of the low-pass coefficients to calculate the “local” trend. Let $x(t_k)$ be a time series type.

**Figure 1.** (Left) Frequency spectrum splitting by the filters $h[n]$ and $g[n]$. (Right) The structure of a three-level fast wavelet transform.
of data, where \( t_k = k \Delta t \) and \( k = 1, 2, \ldots, N \). The algorithm that we employ involves the following steps:

1. Determine the profile \( Y(k) = \sum_{i=1}^{k} (x(t_i) - \langle x \rangle) \) of the time series, which is the cumulative sum of the series from which the series mean value is subtracted.

2. Compute the fast wavelet transform (FWT), i.e., the multilevel wavelet decomposition of the profile. For each level \( m \), we get the fluctuations of the \( Y(k) \) by subtracting the “local” trend of the \( Y \) data, i.e., \( \Delta Y(k; m) = Y(k) - \hat{Y}(k; m) \), where \( \hat{Y}(k; m) \) is the reconstructed profile after removal of successive details coefficients at each level \( m \). These fluctuations at level \( m \) are subdivided into windows, i.e., into \( M_s = \text{int}(N/s) \) non-overlapping segments of length \( s \). This division is performed starting from both the beginning and the end of the fluctuations series (i.e., one has \( 2M_s \) segments). Next, one calculates the local variances associated to each window \( \nu \).
Figure 3. Same plots as in Fig. 2 but for rule 150.

\[ \begin{align*}
F^2(\nu; s; m) &= \text{var} \Delta Y((\nu - 1)s + j; m), \\
\text{for } j = 1, \ldots, s, \nu = 1, \ldots, 2M_s, \quad M_s = \text{int}(N/s). \tag{8}
\end{align*} \]

1. Calculate a \( q \)-th order fluctuation function defined as

\[ F_q(s; m) = \left( \frac{1}{2M_s} \sum_{\nu=1}^{2M_s} |F^2(\nu, s; m)|^{q/2} \right)^{1/q} \tag{9} \]

2. As in Ref. 7 and 21.

3. In order to determine if the analyzed time series have a fractal scaling behavior, the fluctuation function \( F_q(s; m) \) should reveal a power law scaling

\[ F_q(s; m) \sim s^{h(q)}, \tag{10} \]

where \( h(q) \) is called the generalized Hurst exponent [21] since it can depend on \( q \), while the original Hurst exponent is \( h(2) \). If \( h \) is constant for all \( q \) then the time series is monofractal, otherwise it has a MF behavior. In the latter case, one can calculate various other MF scaling exponents, such as \( \tau(q) \) and \( f(\alpha) \) [10].

4. Application to ECA

We apply the previous algorithm to the time series of two illustrative ECA as classified by Wolfram in 1984 [22].
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Figure 4. (a) Time series of the row signal of $H_{511}$. Only the first 128 points are shown of the whole set of $2^{10} - 1$ data points. (b) Profile of the row signal of $H_N$. (c) Generalized Hurst exponent $h(q)$, (d) the $\tau(q)$ exponent, and (e) the singularity spectrum $f(\alpha)$.

Consider the time series of the so-called row sum ECA signals, i.e., the sum of ones in sequences of rows, employing the db-4 Daubechies wavelet function. We have found that a better matching of the results given by the WMF-DFA method with those of other methods is provided with this wavelet. In Figs. (2-4) are illustrated the results for the rule 90 and 150, respectively, when the first row is all 0s with a 1 in the center. The fact that the generalized Hurst exponent is not a constant horizontal line is indicative of a multifractal behavior in all three cases. In addition, the fact that the $\tau$ index is not of a single slope is another clear feature of multifractality.

The strength of the multifractality is roughly measured with the width $\Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}}$ of the parabolic singularity spectrum $f(\alpha)$ on the $\alpha$ axis. For example, for the impulsive initial condition, $\Delta \alpha_{90} = 0.9998(1.0132)$, and $\Delta \alpha_{150} = 1.011(1.0075)$, when the MF-DFA (WMF-DFA) are employed. We notice that the most “frequent” singularity for all the analyzed time series occurs at $\alpha = 0.568$, where the width $\Delta \alpha$ of rule 90 is shifted to the right with respect to those of 150. According to our results, the strongest singularity, $\alpha_{\text{min}}$, of all time series corresponds to the rule 90 and the weakest singularity, $\alpha_{\text{max}}$, to the rule 150. In Ref. 14 are shown the results for different initial center pulses for rules 90, 105, and 150.
With the aim of computing the pseudo random sequences of $N$ bits, in Ref. 18 has considered an algorithm based on the backward evolution of the CA rule 90. Here, we analyze the time series of the row sums of the sequence matrix $H_N$, which was used here to generate recursively the pseudo random sequences. This matrix has dimensions $(2^N + 1) \times (2^N + 1)$. The results for the row sums of $H_{511}$ are illustrated in Fig. 4. Although the profile is different, the results are similar with those obtained for the rule 90, see Fig. 2. A more complete analysis of this matrix is carried out in Ref. 23.

5. Conclusions

We have analyzed the time series of the so-called row sums ECA signals, with an algorithm that integrates the discrete wavelet transform in the MF-DFA technique. This algorithm has shown to be a well-suited procedure to analyze the multifractal properties of the ECA. Since the evolution of the sequence matrix $H_N$ is based on the evolution of the CA rule 90, a multifractal structure of the entries of $H_N$ is revealed and quantified according to the multifractal formalism. Indeed, we get similar results to the other methods but computationally faster because it employs a lesser number of windows. In addition, our results represent a confirmation of the fact that ECA have intrinsic multifractality that does not depend on the set of initial data that we used. Therefore, when processes thought to be multifractal are simulated with ECA, their intrinsic multifractal behavior should be taken into account as a feature of the simulation procedure rather than of the multifractal behavior of the simulated processes.

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