Dynamics of an irreversible cooling cycle

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A local stability analysis of an irreversible energy converter working in several optimum regimes, is presented. The energy converter in the present work is a cooling cycle that exchanges heat with the heat reservoirs at $T_1$ and $T_2$ ($T_1 > T_2$) through a couple of thermal conductors, with the same conductance value ($\beta$). The working fluid has a constant heat capacity ($C$) in the two isothermal branches of the cycle. From the local stability analysis at the maximum COP and maximum ecological function regimes, we observe thermodynamic and dynamical properties similar to those found for other thermal cycles studied by several authors. The most important thermodynamic property that we find is that this converter also has optimal points of operation (maximum COP and maximum ecological function). On the other hand, with respect to the dynamic properties we find that under small perturbations this converter shows a robust behavior, i.e. the converter returns quickly to the steady state corresponding to one of the optimal points of operation. In the case of maximum COP the relaxation times are independent of the external temperatures of the cooling cycle but not of the internal irreversibilities, whereas in the case of maximum ecological function these times depend as much on the external temperatures as of the internal irreversibilities. We conclude from the assertions in the previous paragraph that the energetic properties of the energy converter at the maximum ecological function regime worsen as $\eta_C$ (Carnot efficiency) decreases. Moreover, since both relaxation times decrease as $\eta_C$ decreases, the decrements of $\eta_C$ improve the system stability and deteriorate its thermodynamic properties. Finally we can say that, $\eta_C$ drives a trade-off between stronger stability (dynamic robustness) and better thermodynamic properties like cooling power and COP.

Keywords: Non-equilibrium and irreversible thermodynamics; Performance characteristics of energy conversion systems; figure of merit.

Descriptores: Termodinámica irreversible y de no–equilibrio; Desempeño de convertidores de energía; Optimización.

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1. Introduction

It is well known that the limits imposed by Classical Equilibrium Thermodynamics (CET) are obtained by the consideration of reversible processes [1–3]. This represents a limitation for CET and leads to process variables too far from measured real values reported in literature. This limitation has been able to be overcome in a great part with the so called Finite Time Thermodynamics (FTT) initiated...
by Curzon-Ahlborn among other researchers [4–6]. Most of
the realized studies in FTT have been concerned with the
energetic properties of steady-states, which are important
from the design point of view, but only a few of them had taken into
account the system’s dynamic properties, like responses due
to noise perturbations or system’s steady-state stability, see
and Páez-Hernández R.T. et al [7–9]. In cooling cycles it has
been observed that considering the internal irreversibilities is
an essential step [10], for any model attempting to describe
the energetic performance of refrigerators for obtaining re-
results reasonably close to those experimentally obtained. Also
the external irreversibilities in such engines play a relevant
role since the materials and sizes of the refrigerators are re-
lected on them. Such as in the power cycles the internal irre-
versibilities have been modeled in many ways. Gordon J.M.
et al [11] point out that the most reliable characteristic curves
are obtained, when in the Clausius equation non compensated
heat term is directly considered. The non-equilibrium models
that have been developed from this form of the Clausius equation
have the advantage of including information (character-
istic functions in several operation modes) that allows extra-
thermodynamic analysis of the dynamic properties of totally
irreversible thermal cycles by using the local stability the-
ory. In the FTT literature of cooling cycles, the performance
optimization for Carnot cooling cycles, vapor-compression
cooling cycles, Stirling cooling cycles, absorption cooling
cycles and thermoelectric cooling cycles have been carried
out. Many results have been obtained, see for example: Be-
jan A. [12], Berry R.S. et al [13], Chen L. et al [14], Chen
L. et al [15], Chen L. [16]. Recently, some progress in the
research and development of air Brayton refrigeration cycles
has been achieved, for example; Mc Cormick J.A. [17], Ham-
lin S. et al [18]. The method of FTT analysis has been applied to
the performance studies of simple and irreversible Brayton
refrigeration cycles (see [19, 20]). The paper is organized as
follows: in section II we describe the cooling cycle’s char-
acteristic energetic functions and some of the well known
steady-state general properties; in section III we establish the
local stability analysis of this irreversible thermal cycle in
general. In section IV we apply this theory to the cooling
cycle at maximum COP regime and at maximum ecological
function regime. Finally, in section V we make some con-
cluding remarks.

2. Steady-state of the irreversible cooling cycle

In this section we introduce some results concerning the ir-
reversible cooling cycle, with the purpose to establish the
basis of the dynamical analysis of this cycle in two differ-
ent regimes of operation, the maximum COP regime and the
maximum ecological function regime.

2.1. Characteristic functions

The cooling cycle is shown in figure 1. The engine operates
between temperatures $T_2$ and $T_1$, with $T_1 > T_2$, the heat
flows irreversibly from $T_2$ to $T_{2t}$ and from $T_{1t}$ to $T_1$
($T_{2t} < T_2 < T_1 < T_{1t}$).

Starting from figure 1, the universe entropy production
can be stated as,

$$\sigma_U = \frac{(1 - \eta_C) Q_a}{T_2} - \frac{Q_c}{T_{2t}} - \frac{Q_c}{T_2} + \beta r > 0, \quad (1)$$

where, $\sigma_U$ is the universe’s entropy production, $Q_c$ is the heat
flux from $T_2$ to $T_{2t}$, $Q_a$ is the heat flux from $T_{1t}$ to $T_1$, $r$
is the parameter that measures the internal irreversibilities,
$\eta_C = 1 - \frac{T_2}{T_{2t}}$ and $\beta$ the thermal conductance. We will con-
sider that the working fluid works in cycles. We obtain the
working fluid’s entropy as

$$\sigma_w = -\frac{Q_a}{T_{1t}} + \frac{Q_c}{T_{2t}} + \beta r = 0. \quad (2)$$

Then the entropy change for this engine is given by

$$\Delta S = \frac{(1 - \eta_C) Q_a}{T_2} - \frac{Q_c}{T_2} + \beta r > 0, \quad (3)$$

In this work we consider that the heat transfer law in steady-
state is given by the Newton heat transfer law, so in the
steady–state for the fluxes of heat showed in figure 1, we ob-
tain

$$J_a = Q_a = \beta \frac{T_2}{1 - \eta_C} \left[ \frac{(1 - \eta_C) T_{1t}}{T_2} - 1 \right] \quad (4)$$

and

$$J_c = Q_c = \beta T_2 \left( 1 - \frac{T_{2t}}{T_2} \right). \quad (5)$$

Here and thereafter the bar means steady-state quantity.
On the other hand, solving equation (2) for \( T_{1t} = T_{1t}(T_{2t}, \beta, \eta_C, r) \) and using the first law of thermodynamics, the power input of the cooling cycle is given by,

\[
P(\beta, \eta_C, T_2, T_{2t}, r) = \beta \left( \left( \eta_C - 2 \right) T_{2t}^2 + 2T_2 \left[ -2r - (3 - r)\eta_C + 4 \right] T_2 - (2 - r)T_{2t}^2 (1 - \eta_C) \right) / \left( T_2 - (2 - r)T_{2t} \right) (1 - \eta_C),
\]

and the COP of the cycle is,

\[
\varepsilon(\eta_C, T_2, T_{2t}, r) = \frac{(T_2 - T_{2t}) (T_2 - (2 - r)T_{2t}) (1 - \eta_C)}{(\eta_C - 2) T_{2t}^2 + 2T_2 \left[ -2r - (3 - r)\eta_C + 4 \right] T_2 - (2 - r)T_{2t}^2 (1 - \eta_C)}.
\]

Finally, if we consider again the figure 1. The steady-state values of the heat fluxes \( J_a \) and \( J_c \) for an arbitrary working regime \( M\phi \), according to the first and second laws of Thermodynamics are given by

\[
J_c = T_{2t}^{M\phi} \left[ \mathcal{P} \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) - r\beta T_{1t}^{M\phi} \right] / T_{1t}^{M\phi} - T_{2t}^{M\phi},
\]

and

\[
J_a = T_{1t}^{M\phi} \left[ \mathcal{P} \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) - r\beta T_{2t}^{M\phi} \right] / T_{1t}^{M\phi} - T_{2t}^{M\phi}.
\]

In next section we show the basis of dynamic analysis of the cooling cycle under small perturbations.

### 3. Dynamics of the irreversible cooling cycle

The reservoirs \( T_{1t} \) and \( T_{2t} \) are not real heat reservoirs but macroscopic objects with a heat capacity \( C \) (working fluid). Therefore, under external perturbations the temperatures of the cooling cycle changes according of the following equations

\[
\frac{dT_{1t}^{M\phi}}{dt} = \frac{1}{C} \left[ J_a \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) - Q_a \right],
\]

and

\[
\frac{dT_{2t}^{M\phi}}{dt} = \frac{1}{C} \left[ Q_c - J_c \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) \right],
\]

where \( J_a \) and \( J_c \) are the non steady-state values of the heat fluxes from the working cycle to the reservoirs. Both of these derivatives vanish when \( T_{1t} \) and \( T_{2t} \) take their steady-state values.

#### 3.1. Local stability analysis

Let \( f \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) \) and \( g \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) \) the following functions [21]:

\[
\begin{pmatrix}
\frac{\partial f}{\partial T_{1t}^{M\phi}} - z & \frac{\partial f}{\partial T_{2t}^{M\phi}} - z \\
\frac{\partial g}{\partial T_{1t}^{M\phi}} & \frac{\partial g}{\partial T_{2t}^{M\phi}}
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

Now in the above equations, \( P \left( T_{1t}^{M\phi}, T_{2t}^{M\phi} \right) \) is the power input of the cooling cycle in the perturbed state. If \( T_{1t} \) and \( T_{2t} \) are very close to the steady-state, we can write \( T_{1t}(t) = T_{1t} + \delta T_{1t}(t) \) and \( T_{2t}(t) = T_{2t} + \delta T_{2t}(t) \), where \( \delta T_{1t}(t) \) and \( \delta T_{2t}(t) \) are small perturbations. By substitution into equation (12) and equation (13), and by using the smallness of \( \delta T_{1t}(t) \) and \( \delta T_{2t}(t) \) we can make a first order approximation and obtain the following differential equations for \( \delta T_{1t}(t) \) and \( \delta T_{2t}(t) \):

\[
\frac{d\delta T_{1t}(t)}{dt} = \frac{\partial f}{\partial T_{1t}} \delta T_{1t}(t) + \frac{\partial f}{\partial T_{2t}} \delta T_{2t}(t)
\]

and

\[
\frac{d\delta T_{2t}(t)}{dt} = \frac{\partial g}{\partial T_{1t}} \delta T_{1t}(t) + \frac{\partial g}{\partial T_{2t}} \delta T_{2t}(t)
\]

Assuming that \( \delta T_{1t}(t) \) and \( \delta T_{2t}(t) \) are of the form \( \delta T_{1t}(t) = ae^{zt} \) and \( \delta T_{2t}(t) = be^{zt} \), with \( z \) a complex number that will be determined. Substituting these equations into equations (14) and (15) leads to the following set of homogeneous linear algebraic equations for \( a \) and \( b \):

\[
\begin{pmatrix}
\frac{\partial f}{\partial T_{1t}} - z & \frac{\partial f}{\partial T_{2t}} - z \\
\frac{\partial g}{\partial T_{1t}} & \frac{\partial g}{\partial T_{2t}}
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
This system of equations has non-trivial solutions only if the determinant of the matrix of coefficients equals zero, i.e.

$$\left(\frac{\partial f}{\partial T_{1t}} - z\right)\left(\frac{\partial g}{\partial T_{2t}} - z\right) - \frac{\partial f}{\partial T_{2t}} \frac{\partial g}{\partial T_{1t}} = 0 \quad (17)$$

This equation is called the characteristic equation. Then the only possible solutions of equations (16) are those with a $z$ which is the solution of the characteristic equation (17). We will denote the solutions of equation (17) as $z_1$ and $z_2$ and by a numerical calculation we find that both results are real and negative. Then from $\delta T_{1t}(t)$ and $\delta T_{2t}(t)$ we see that any perturbation decays exponentially with time. The relaxation time depends on the absolute value of $z_1$ and $z_2$. Indeed, we can define two relaxation times $\lambda_1$ and $\lambda_2$ as [21]

$$\left(\frac{\lambda_1}{\lambda_2}\right) = -\left(\frac{1}{z_2}\right). \quad (18)$$

4. Irreversible cooling cycle at the maxima cop and ecological function regimes

4.1. Stability at maximum COP

The power input needed for the irreversible cooling cycle in steady-state depends on the stationary temperatures ($T_{1t}^{M\phi}$ and $T_{2t}^{M\phi}$). We are interested in studying the dynamic properties of this engine when it deviates from the steady-state. For this end we need to know how the power ($P\left(T_{1t}^{M\phi}, T_{2t}^{M\phi}\right)$)

depends on the temperatures in a non-steady-state. Furthermore, only small deviations from the steady-state are considered in a local stability analysis, which is the aim in this work. Therefore, it seems reasonable to assume that the power input needed by the irreversible cooling cycle is related to the temperatures in a non-stationary state in the same way that in steady-state, so after substituting equations (4) and (5) into equation (6) and using the low temperature of the working regime of maximum COP: $T_{2t}^{M\epsilon}$ shown in figure 2a for particular values of $r$, we obtain the power input of the cooling cycle at maximum COP regime as,

$$P_{ME}\left(\beta, \eta_C, T_2, r\right) = \beta T_2 \left(2(2 - r)r + \left(\sqrt{2 - r} - (2 - r)r\right)\eta_C\right) \quad (19)$$

We apply the stability theory of section III to the irreversible cooling cycle, for finding the relaxation times as function of the Carnot efficiency $\eta_C$ in this working regime. For this regime the characteristic equation roots’ are numerically calculated and we find that, both are real and negative. We show the plot for the relaxation times in the figure 3a. In this figure we observe that $\lambda_1^{ME}$ and $\lambda_2^{ME}$ are constants, then we can suppose that the engine is dynamically robust for any value of $\beta$, and $C$, after small perturbations.

4.2. Stability at maximum ecological function

Considering now the ecological regime proposed by Angulo-Brown [22], which consists in optimizing the function,

$$E = P - T_2 \sigma_T, \quad (20)$$

that represents an austere compromise between power output and wasted energy of the engine [23, 24], $E$ is the so-called ecological function, $P$ is the power output, $T_2$ is the cold reservoir temperature and $\sigma_T$ the entropy production. Figure 2b shows the ecological function as function of the low temperature $T_{2t}$ and the internal irreversibilities $r$. Following step by step the procedure shown in section III and the above subsection, with the exception that the expression for the power input of the cooling cycle at maximum ecological function is,

$$P_{ME}\left(\beta, \eta_C, r\right) = \beta T_2 \left(2r + \left(-r - \frac{2}{\sqrt{2 - r}} + 1\right)\eta_C + \frac{3}{\sqrt{2 - r}} - 2\right)\quad (21)$$

We find that $z_1$ and $z_2$, as in the above regime, are real and negative. The figure 3b shows the plot of the relaxation times $\lambda_1^{ME}$ and $\lambda_2^{ME}$ versus $\tau = \frac{T_2}{T_1}$ in the maximum ecological function working regime. Finally, in figure 4 we show a qualitative phase portrait for this system, at the working regimes studied in this articles. The figure 4 shows the slow and fast directions, for which the cooling cycle returns to equilibrium. In both modes of operation, if the system is perturbed so that it falls into the first quadrant, immediately return to the steady state, whereas if the perturbed system falls into the other quadrants its return to the steady state will be a little slower.

5. Concluding Remarks

The cooling cycle model proposed in this work was analyzed under two regimes of performance: maximum COP and maximum ecological function: In the first case the conditions are maximum cooling power and minimum energy input. In the second one, the condition is a good trade-off between high cooling power and high COP. The cooling cycle under these two working regimes is dynamically robust, that is, it is locally stable under small perturbations. The both mentioned regimes are steady-state processes. For the maximum COP regime, in figure 2a can be observed that the relaxation times only depend on the internal irreversibilities. In figure 4a (phase portraits) it is shown how the cooling cycle under small perturbations rapidly returns to its steady-state. For maximum ecological function regime the relaxation times depend on both the external temperatures and the internal irreversibilities (see figure 2b). However, this regime is also locally stable and the cooling cycle rapidly recovers its steady state (see figure 4b). In summary, this cooling cycle is locally stable under small perturbation, in the same way that many other steady-state thermal cycles reported in the literature. This local stability is a necessary condition for the energy converter operation under optimization criteria as those mentioned in this work.

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