Particle Motion in Static Homogeneous Electric and Magnetic Fields and its Consequences in a Typical Confinement Situation in Plasma Physics

G. Ares de Parga, R. Mares, and M. Ortiz-Domínguez

Received 30 de junio de 2011; accepted 25 de agosto de 2011

A simple representation of the Landau-Lifshitz equation of motion for a charged particle is deduced. As a consequence of the new representation, a simple technique is developed in order to solve the equation. As an example of it, the solutions for the electric and magnetic constant field cases are found for a typical confinement in Plasma Physics. A discussion about the validity of the relativistic Larmor formula is exposed.

Keywords: Reaction force; radiated power; field theory; plasma physics.

1. Introduction

It is a well-known fact that during the process of confinement of a plasma a certain number of instabilities due to many reasons appears [1]. Hakim and Mangeney [2] have included the effect of the reaction force in the calculation of the relativistic kinetic equations in a plasma. Although they obtained a modification of the Vlasov equation, their result was oriented to understand the irreversibility of the modified equation. However, in order to include the reaction force in the theory, they used the Landau-Lifshitz equation of motion for a charged particle [3,4]. In those times, this proposal was not well-known and consequently it was not sufficiently recognized. Indeed, the classical Lorentz-Dirac equation [5] presented so many physical contradictions that normally everything was analyzed by using the regular Lorentz equation of motion. However, in Plasma Physics, the consequences of considering the changes in the trajectories of the charges due to the reaction force have not been analyzed enough. On the other hand, during the seventies, Shen [6] showed that the reaction force must be taken into account when the energies and fields belong to a certain constraint zone, the so-called Shen’s zone where quantum effects may be neglected. The energies and fields that appear in Plasma Physics belong to Shen’s zone. Other constraints about the use of the reaction force has been analyzed by Hammond [7]. Therefore, in Plasma Physics, the motion of a charge must be analyzed by considering a reaction force. Nowadays, the Landau-Lifshitz [LL] equation is considered, by many authors [8-10,4,11], as the equation which describes the motion of a spinless charged point particle. Indeed, by introducing a simple approximation to the Lorentz-Dirac [LD] equation [5] for point charged particles in classical electrodynamics, Landau and Lifshitz [3] obtained an equation which has been mathematically supported by Spohn [8] later on. Rohrlich [4] physically has reinforced the goodness of the LL equation by noticing that it is a second order differential equation which neither presents runaway solutions nor preaccelerations. It is very important to note that even if we can obtain the LL equation by substituting the Lorentz-Dirac equation in the LD reaction term and consequently it could be considered as an approximation or a first order expansion in \( \tau_\alpha (\tau_\alpha = (2/3)(q^2/mc^3) \) is called the characteristic time), for Spohn [8] and Rohrlich [4] the LD equation must be restricted to its critical surface yielding the LL equation. Moreover, Spohn [8] assures that the approximations done in order to obtain the LL equation of motion are similar to the ones done in order to deduce the LD equation. Due to all these reasons and the fact that LL equation is a second order differential equation which does not present unphysical solutions as the runaway solutions and the preaccelerations, Spohn [8] and Rohrlich [4] concluded that the LL equation represents the correct equation of motion for a spinless classical point charge [4]. Although there exist some works against the correctness of the LL equation [12,13], the LL equation of motion seems to represent the best choice to describe the motion of a charge in Plasma Physics.

The objective of the present paper is threefold: first, we obtain a new representation of the LL equation in Sec. 2; second, in Sec. 3, we present a better representation of the LL equation which permits to develop a simple technique to solve in some particular cases the LL equation. In fact, we give the solutions for the constant electric and magnetic fields and we show that an instability in the confinement of a Plasma may appear due to the reaction force. Finally, in Sec. 4, due to some physical problems that appear related with the radiated power [15,9,16], we present some interest-
ing aspects about the Larmor formula and we justify the technique to obtain the LL equation in Sec. 2. Some concluding remarks are exposed in Sec. 5.

2. New representation of the Landau-Lifshitz equation

Some authors express the LD equation of motion as

\[
ma^{\mu} = F^{\mu} + \tau_o m \left[ \frac{\alpha^{\mu}}{\alpha} + \frac{a^2}{c^2} v^{\mu} \right],
\]

(1)

where \( F^{\mu} \), \( \tau_o \), \( m \) and \( c \) represent the applied force, the characteristic time of the charge \( \tau_o = (2/3)(q^2/mc^3) \), the mass of the charge and the speed of light, respectively. \( \alpha^{\mu} = (d\alpha^{\mu}/d\tau) \) and \( v^{\mu} \) represent the 4-acceleration and the 4-velocity of the charge that satisfy Eq. (1). However, since a new proposal for the radiated power will be discussed in Sec. 4, we will just consider the LD equation for the electromagnetic case; that is:

\[
ma^{\mu} = (q/c)F^{\mu\nu}v_\nu + \tau_o m \left[ \frac{\alpha^{\mu}}{\alpha} + \frac{a^2}{c^2} v^{\mu} \right],
\]

(2)

where \( F^{\mu\nu} \) the field-strength tensor. On the other hand, in order to obtain the LL equation let us analyze some important aspects for the deduction of the reaction term. First of all, we have to notice Hammond’s comment [7]: “In other words, it is not assumed a priori that a self-interactions exists, it is not put into the action principle from which the equations are derived, it is a consequence of the theory. This result, that self-interactions are not included in the basic formulation of theory, has been called a formal inconsistency in the theory”. In other words, this force is not a part of the theory, but is put in by hand. Therefore, any deduction of the reaction force is based on an external assumption to the theory. Indeed, let us consider the L equation,

\[
ma^{\mu} = (q/c)F^{\mu\nu}v_\nu.
\]

(3)

By using Maxwell equations and the L equation [14], we can define the Poynting vector and consequently obtain the power radiated by a charged particle, that is, the Larmor formula. In a heuristic form and by assuming a periodic motion, some deductions of the Lorentz-Abraham equation are based on the Larmor formula [14]. Then, by including the principles of Special Relativity, the LD equation is deduced. Moreover, by defining the relativistic Maxwell stress tensor, Dirac [5] obtained the LD equation of motion by assuming the principle of simplicity and the renormalization of the mass. In both cases, we begin with an equation of motion for a charged particle, the L equation, and by making a physical assumption we end up with another equation, the LD equation. That is, some physical assumptions have been done in order to obtain the desired equation. On the other hand, in order to obtain the LL equation, Spohn [8] started from the LD equation and used a formal perturbation theory. Once again, an assumption has been done to deduce a reaction force.

Our deduction of the LL equation will be based on the following assumption: since the LD reaction term is obtained by starting with the L force, we assume that the LD radiation term is just valid when the particle follows the L equation [9,16]. That is, a constraint has been imposed on the motion of the particle to compensate the LD reaction force, in order the charge moves satisfying the L equation. Therefore, there is no contradiction since the motion will be described in this case by the L equation and not by two equations as we notice before. However, when the charged particle is not obligated to move satisfying Eq. (3), the L equation, no constraint is imposed, we can suppose that the motion of the charged particle must be described by another equation where a reaction term must be included. Finally, our main proposal consists of considering that the reaction force is formally equal to the LD reaction force but with the difference that the acceleration included on it corresponds to the acceleration due to the L equation. That is, in the reaction term of the LD equation,

\[
ma^{\mu} = (q/c)F^{\mu\nu}v_\nu + \tau o m \left[ \frac{\alpha^{\mu}}{\alpha} + \frac{a^2}{c^2} v^{\mu} \right],
\]

where \( \alpha^{\mu} \) and \( \alpha \) must be substituted by the acceleration due to the L force. The result is

\[
ma^{\mu} = \frac{q}{c} F^{\mu\nu} v_\nu + \tau o \left[ \frac{d}{d\tau L} \left( \frac{q}{c} F^{\mu\nu} v_\nu \right) + \frac{q^2}{c^2 m} F^{(2\nu)} \right],
\]

(4)

where \( F^\nu = F^{\mu\beta} v_\beta, \ F^2 = F^{\mu\nu} F_\nu = F^{\mu\beta} v_\beta F_{\eta\gamma} v^\eta v^\gamma \) and \( \tau L \) represents an invariant quantity defined in each point of the real trajectory of the particle which coincides with the proper time of a virtual charged particle which moves satisfying the L equation with the same 4-velocity \( v^\mu \) at the crossing point between the real and L trajectories. That is:

\[
\frac{d\alpha^{\mu}}{d\tau L} = \frac{q}{c} F^{\mu\nu} v_\nu.
\]

(5)

Therefore the L-time, \( \tau L \), is different from the real proper time of the charged particle and it is just used in order to calculate the term

\[
\frac{dF^{\mu\nu}}{d\tau L} \left[ \frac{q}{c} F^{\mu\nu} v_\nu \right]
\]

of Eq. (4) which represents the main point to deduce the LL equation (see Fig 1). We also obtain,

\[
\frac{dF^{\mu\nu}}{d\tau L} = \partial F^{\mu\nu} \frac{dx^\alpha}{d\tau L}.
\]

(6)

Since by definition of the L-time, \( \tau L \), the 4-velocity \( v^\mu \) coincides with \( dx^\mu /d\tau L \) in each point of the trajectory of the charge, we have

\[
\frac{dF^{\mu\nu}}{d\tau L} = \partial F^{\mu\nu} \partial x^\alpha v^\alpha.
\]

(7)

Then,

\[
\tau o \frac{d}{d\tau L} \left( \frac{q}{c} F^{\mu\nu} v_\nu \right) = \tau o \left[ \frac{dF^{\mu\nu}}{d\tau L} \partial x^\alpha v^\alpha + F^{\mu\nu} \frac{dv_\nu}{d\tau L} \right].
\]

(8)
By using Eq. (5), we obtain
\[ \tau_o \frac{d}{d\tau_L} \left[ \frac{q}{c} F^{\mu\nu} v_\nu \right] = \frac{q}{c} \left[ \frac{\partial F^{\mu\nu}}{\partial x^\alpha} v^\alpha v_\nu + \frac{q}{cm} F^{\mu\nu} F_{\alpha\nu} v^\alpha \right]. \] (9)

Finally, due to the antisymmetric property of the field-strength tensor \( F^{\mu\nu} \), we arrive at
\[ \tau_o \frac{d}{d\tau_L} \left[ \frac{q}{c} F^{\mu\nu} v_\nu \right] = \tau_o \frac{q}{c} \left[ \frac{\partial F^{\mu\nu}}{\partial x^\alpha} v^\alpha v_\nu - \frac{q}{cm} F^{\mu\nu} F_{\alpha\nu} v^\alpha \right]. \] (10)

Introducing this last result in Eq. (4), we obtain the LL equation of motion,
\[ m a^\mu = \left( \frac{q}{c} \right) F^{\mu\nu} v_\nu + \tau_o \frac{q}{c} \left[ \frac{\partial F^{\mu\nu}}{\partial x^\alpha} v^\alpha v_\nu - \frac{q}{cm} F^{\mu\nu} F_{\alpha\nu} v^\alpha \right] - \left( \frac{q}{cm} \right) F^{\mu\nu} F_{\alpha\nu} v^\alpha + \left( \frac{q^2}{c^4 m} \right) F^2 v^\mu. \] (11)

This last equation, the LL equation, seems very difficult to be solved. However, Eq. (4) represents the same equation than the LL equation, Eq. (11), but it is presented in such a manner that it is simpler to be solved. Moreover, starting from Eq. (4), another representation of the LL equation can be obtained in order to simplify the technique to obtain the solutions of it. The LL equation has been studied by many authors but the work done by Griffiths [13] is very interesting since it made a comparison between the Lorentz-Abraham and the non relativistic LL equations.

3. Simplified representation of the LL equation

By defining the constant \( w = \frac{q}{cm} \), Eq. (4) can be rewritten as
\[ a^\mu = w F^{\mu\nu} v_\nu + \tau_o \frac{d}{d\tau_L} \left[ w F^{\mu\nu} v_\nu \right] + \tau_o \frac{w^2}{c^2} F^2 v^\mu. \] (12)

In order to develop a technique to solve the LL equation of motion in a simpler form, we will define two 4-vector fields in each point of the trajectory of the particle. Indeed, as for the L-time, \( \tau_L \), by using Eqs. (3) and (10), we can define two 4-vector fields \( a_L^\mu \) and \( d a_L^\mu/d\tau_L \), in the following way:
\[ a_L^\mu = \frac{q}{cm} F^{\mu\nu} v_\nu, \] (13)
and
\[ \frac{d a_L^\mu}{d\tau_L} = \frac{d}{d\tau_L} \left[ \frac{q}{c} F^{\mu\nu} v_\nu \right] = \frac{q}{c} \left[ \frac{\partial F^{\mu\nu}}{\partial x^\alpha} v^\alpha v_\nu - \frac{q}{cm} F^{\mu\nu} F_{\alpha\nu} v^\alpha \right]. \] (14)

The fields represent the acceleration and the rate of acceleration with respect the proper time of a charged particle that moves following the trajectory generated by the L force in each point of the real trajectory of the charge (see Fig. 1). Substituting this fields in Eq. (12), we obtain the following equation,
\[ a^\mu = a_L^\mu + \tau_o \left[ \frac{d a_L^\mu}{d\tau_L} + \frac{w^2}{c^2} F^2 v^\mu \right]. \] (15)

This last equation is also a representation of the LL equation and it is the desired equivalent form of the LL equation. It has to be pointed out that the LL equation has the same form of the LD equation, but with \( a^\mu \) substituted by \( a_L^\mu \) in the right member of the equation. Therefore, in order to solve the LL equation it is necessary to firstly analyze the L equation which represents a strong simplification. Indeed, it has to be noticed that Eq. (15) is a second order equation because the terms \( d a_L^\mu/d\tau_L \) is just a 4-vector field which depends on the 4-vector position \( x^\mu \) and the 4-vector velocity \( v^\mu \) of the charge. Consequently, since we have noticed its equivalence with the LL equation, runaway solutions and preaccelerations are avoided. The interesting property of this representation consists of not having to calculate all the terms that appears in the regular expression of the LL equation and this point will be showed in the next subsections.

3.1. The constant electric field

In order to apply our new representation of the LL equation, we want to discuss the simple case of one charged particle in a constant electric field with intensity given by \( E_o \). Let us constrain the motion and the field in the \( x \)-axis. We begin by solving the L equation for this case. We obtain the two L equations,
\[ a_L^0 = \frac{d^2 ct}{d\tau_L^2} = w E_o \frac{d}{d\tau_L} x \]
\[
\alpha_L^2 = \frac{d^2}{d\tau_L^2} x = wE_0 c\frac{d}{d\tau_L} t, \tag{16}
\]

By using the last result, \(\alpha_L^2\) is

\[
\alpha_L^2 = \left( c \frac{d^2}{d\tau_L^2} + \left( \frac{d^2}{d\tau_L^2} \right)^2 \right) x = -w^2 E_0^2 c^2. \tag{17}
\]

Then,

\[
\frac{a^2_L}{c^2} = -w^2 E_0^2. \tag{18}
\]

On the other side,

\[
\frac{d}{d\tau_L} a_L^0 = wE_0 \frac{d^2}{d\tau_L^2} x = w^2 E_0^2 c t \tag{19}
\]

\[
\frac{d}{d\tau_L} a_L^x = wE_0 \frac{d^2}{d\tau_L^2} c t = w^2 E_0^2 x, \tag{20}
\]

where the point “\(\cdot\)” represents the real proper time derivative \((d/d\tau)\) since by definition \(dx^\mu/d\tau_L\) coincides with the 4-velocity \(u^\mu\) at each point of the real trajectory. Therefore,

\[
\tau_o \left[ \frac{d\mu}{d\tau_L} + \frac{a^2_L}{c^2} u^\mu \right] = 0. \tag{21}
\]

Since the LL reaction term vanishes in this case, constant electric field parallel to the motion, the solution of the LL equation coincides with the one of the L equation. It has to be noticed that for the LD equation, the LD reaction term also vanishes in this case [17,18].

### 3.2. The constant magnetic field

Let us consider a constant magnetic field \(B\) in the \(z\)-axis. Without loosening generality, we can constrain the problem to two dimensions and by putting \(\Omega = qB/cm\) the three components of the L equation of motion are

\[
\frac{d^2}{d\tau_L^2} x = \Omega y \quad \frac{d^2}{d\tau_L^2} y = -\Omega x. \tag{22}
\]

Therefore,

\[
a_L^2 = -\Omega^2 (x^2 + y^2). \tag{23}
\]

So the LL equation of motion can be expressed by

\[
\dot{c} t = -\tau_o \Omega^2 \frac{1}{c^2} (x^2 + y^2) c t
\]

\[
\dot{x} = \Omega y - \tau_o \Omega^2 x \left[ 1 + \frac{1}{c^2} (x^2 + y^2) \right],
\]

\[
\dot{y} = -\Omega x - \tau_o \Omega^2 y \left[ 1 + \frac{1}{c^2} (x^2 + y^2) \right]. \tag{24}
\]

As we can notice those equations do not depend on the position, \(x\) and \(y\). Therefore, we can define the vectors,

\[
\vec{v} = \vec{x} + \vec{y} = v(\cos \theta \hat{i} + \sin \theta \hat{j}),
\]

\[
\hat{e}_v = \left[ \frac{\partial \vec{v}}{\partial v} \right]^{-1} \frac{\partial \vec{v}}{\partial v} = \cos \theta \hat{i} + \sin \theta \hat{j},
\]

\[
\hat{e}_\theta = \left[ \frac{\partial \vec{v}}{\partial \theta} \right]^{-1} \frac{\partial \vec{v}}{\partial \theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}, \tag{25}
\]

where \(v\) represents the magnitude of the relativistic velocity, that is, \(v = (x^2 + y^2)^{1/2}\), and \(\theta\) is the angle of the vector \(\vec{v}\) with the \(x\)-axis. Eq. (24) can be written as

\[
\left[ \vec{v} \right] \hat{e}_v + \left[ v \theta \right] \hat{e}_\theta = -\tau_o \Omega^2 v (1 + \frac{v^2}{c^2}) \vec{v} - \Omega v \hat{e}_\theta. \tag{26}
\]

Consequently, we can assure that

\[
\dot{\theta} = \Omega = \text{constant}
\]

\[
\frac{1}{v(1 + \frac{v^2}{c^2})} dv = -\tau_o \Omega^2 d\tau. \tag{27}
\]

By integrating the last equation, we arrive at (see Fig. 2):
\[ v = \frac{v_o c \exp -\tau_o \Omega^2 \tau}{[c^2 + v_o^2 [1 - \exp -2\tau_o \Omega^2 \tau]]^{\frac{3}{2}}}. \]  
\[ \hat{v} = \frac{v_o c \exp -\tau_o \Omega^2 \tau}{[c^2 + v_o^2 [1 - \exp -2\tau_o \Omega^2 \tau]]^{\frac{3}{2}}} \times \left[ \cos(\Omega \tau + \delta) \hat{u} + \sin(\Omega \tau + \delta) \hat{j} \right], \]

where \( v_o \) and \( \delta \) represent the speed of the particle and the phase of the trigonometric functions at \( \tau = 0 \), respectively. It has to be noticed that this last solution implies that a drift of the center of motion of the charged particle appears. Moreover, since the drift decays in a few seconds for the electrons, the electrons will loose all the energy and the confinement of the plasma could be affected due to the fact that the electron energy will decay faster than the energy of a proton or a similar ion.

4. New Expression For the Radiated Power and their Consequences

Let us analyze the physical consequences of the LL reaction term. For special situations, as the sudden force case, an infinite energy is deduced when the Larmor formula is used in order to calculate the radiated energy at the time when the sudden force appears. The divergence is obtained when the LL solution is considered and it disappears when LD equation is used. Consequently, Baylis and Hulchilt [15] have proposed a mixed theory which consists of using the LD solution, instead of the LL solution, close to the onset of the sudden force in order to avoid the divergence. Nevertheless, the mixed theory will not provide a general method for other cases: that is, when there is not a sudden force but a soft force, the method does not indicate which one of both, LD or LL solution, should be used. Ares de Parga et al., [9,16] have proposed a new interpretation of the radiated power in order to explain such a problem. Indeed, they consider that the LL equation is not the correct equation but the exact equation which describes the motion of a charged particle within the limits of classical electrodynamics. In this scheme, the reaction term is

\[ \tau_o \left[ \frac{d}{d\tau_L} \frac{q}{c} F^{\mu \nu} v_\nu + \frac{q^2}{c^3 m} F^2 v^0 \right]. \]  

On the other hand, as it was proposed by Rohrlich for the LD equation [21], we can propose the large distance radiated power

\[ P = -\tau_o \frac{q^2}{c^3 m} F^2 v^0, \]  

and the attached radiated power

\[ P_{att} = -\tau_o c \frac{d}{d\tau_L} \left[ \frac{q}{c} F_0 v_\nu \right]. \]

The new expressions for the large distance and the attached radiated powers have been introduced to show that there is not such a divergence for a sudden force when LL equation is used. Therefore, there is a consistence between the radiated power and the LL equation [9]. When the constant electric field case is analyzed, since the reaction term vanishes, it can be thought that there is no radiation. Nevertheless, the large distance radiated power does not vanish. The attached energy provides the energy to the large distance radiated power. This means that there is an arrangement of the energy [9]. The essential idea consists of proposing that the radiation emitted by a point charge is due exclusively to the external exerted electromagnetic forces on the charge. Returning to the simpler expression of the LL reaction term,

\[ \tau_o \left[ \frac{d\phi^\mu}{d\tau_L} + \frac{q^2}{c^2} v^\mu \right], \]

we can express the large distance radiated power described by Eq. (30) as

\[ P = -\tau_o \frac{q^2}{c} v^0. \]

In a similar way, the attached radiated power is

\[ P_{att} = -\tau_o c \frac{d\phi^0}{d\tau_L}. \]

In this model, an interesting fact appears. Indeed, we can conclude that if there is not an electromagnetic field there is no radiation because in the radiated power expression only electromagnetic forces are considered. However, we have noticed at the beginning of Sec. 2 that the LL equation can be used just when electromagnetic forces are applied. We cannot conclude anything about the reaction force and the radiation when other kind of forces are applied. Moreover, there exists a large discussion about the radiation of a charge when it is submitted on a gravitational field. The reader can find a good discussion of this topic in the book of Rohrlich and quote attached [11,22].

5. Concluding remarks

Solutions of the LL equation have been found by using the new representation of the LL equation of motion, Eq. (15). The case of the constant electric field has been already solved by using complicated techniques [18,17] but with the new representation the solution is obtained in a simpler form. The constant magnetic field case is also solved in a simple way by using the same technique. Indeed, for the constant magnetic case, other similar solutions have appeared in the literature [23,24,6] but the technique are complicated and the results are not accurate. The result shows that the particle not only decays but presents a drift of the center of motion. The confinement of a plasma could be affected due to the rapid decay of the energy of the charged particle.

Acknowledgement

This work was partially supported by C.O.F.A.A and E.D.I.,I.P.N, and CONACYT. We thank both referees for valuable and enlightened comments.