General relativistic magnetic monopole

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ABSTRACT. The general relativistic version of the 't Hooft-Polyakov magnetic monopole is studied. We confirm a previously derived result indicating that for a sufficiently large symmetry breaking mass, the monopole is a black hole. The critical mass is evaluated and the existence of an event horizon examined with some detail.

RESUMEN. Presentamos un estudio del monopolo de t’Hooft-Polyakov incluyendo el campo gravitacional. El estudio confirma un resultado obtenido previamente que indica la conversión del monopolo en un hoyo negro para un valor de la masa asociado a la ruptura de simetría suficientemente grande. Calculamos también el valor crítico de dicha masa y examinamos con cierto detalle la existencia del horizonte de eventos.

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1. INTRODUCTION

't Hooft [1] and Polyakov [2] have shown that magnetic monopoles exist in all gauge theories with spontaneously broken symmetry if the symmetry group admits a compact covering. Such monopoles are regular stable solutions of the field equations with finite energy. Later on, Bogomol’nyi and Marinov [3] obtained numerical solutions of the field equations which describe such a monopole.

The inclusion of gravitational effects in gauge field theories has been considered by Van Nieuwenhuizen, Wilkinson and Perry [4], and other authors [5-9]. An interesting problem is the existence of regular solutions of the Einstein-Yang-Mills-Higgs equation, and the possible existence of black holes with 'hairs', that is, parameters other than mass, angular

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momentum and electric charge. It has been found recently that such regular solutions corresponding to a magnetic monopole exist in general relativity, provided the mass for spontaneous symmetry breaking is small; if, on the other hand, this mass is comparable to the Planck mass, the spacetime corresponds to a Reissner-Nordstrom solution with a black hole horizon [10-11].

Given the complexity of the equations and the many parameters involved, we think it is worth to repeat the above mentioned calculations (particularly those of Refs. [10-11]) with a different integration method and a wider range of parameters. In particular, we calculate the norm of the timelike Killing vector. This norm is a particularly important quantity because it defines the horizon of a black hole (it vanishes at the horizon), and has not been evaluated in previous works.

Our model has a SU(2) gauge field with a Higgs field. As pointed out in Ref. [4], gravity does not alter drastically the form of the Yang-Mills and Higgs fields if the mass $v$ related to the vacuum expectation value of the Higgs field is much smaller than the Planck mass; however, black hole solutions exist if the mass $v$ exceeds a critical value [10-11]. Indeed, the numerical solutions for small $v$ are quite similar to those found by Bogomol’nyi and Marinov [3], and are presented in the figures.

The interested readers are referred to Refs. [1-3] for the concept of a magnetic monopole in flat space, and to Refs. [4-11] for the extension of gauge field theories to general relativity.

2. FIELD EQUATIONS

The basic equations were obtained by van Nieuwenhuizen et al. [4]. For the sake of completeness, we outline a slightly different derivation of these equations. The starting point is the Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \langle D_\mu \phi, D_\nu \phi \rangle - V(\|\phi\|) - \frac{1}{4} \langle G_{\mu\nu}, G_{\mu\nu} \rangle \right],$$

(1)

where

$$D_\mu \phi = \partial_\mu \phi - ie [A_\mu, \phi]$$

(2)

and

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu].$$

(3)

Here $\langle A, B \rangle \equiv \text{Trace}(A^\dagger B)$, $\|\phi\|^2 \equiv \langle \phi, \phi \rangle$ and $V$ is the scalar field potential (we set $\hbar = c = 1$).

Now, let the gauge group be SU(2) and set

$$\phi = iQ(r) \frac{x^{(n)}(n)}{r},$$

(4)

$$A^\mu = W(r) \frac{K_{(n)}^{(n)}(n)}{r^2},$$

(5)
where $\sigma_{(n)}$ are the Pauli matrices, $x^{(n)}$ are Cartesian coordinates and $K_{(n)}^\mu$ are the three Killing vectors of the rotation group. Thus, $A^\mu$ is a pure gauge except at $r = 0$.

In general, a static and spherically symmetric metric has the form

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2.$$  \hspace{1cm} (6)

Using this metric, one finds the field equations:

$$e^{-\frac{1}{2}(\nu+\lambda)} \frac{d}{dr} \left[ e^{\frac{1}{2}(\nu-\lambda)} r^2 \frac{dQ}{dr} \right] - 2(1 + 2eW)^2 Q - \frac{1}{2} r^2 \frac{dV}{dQ} = 0,$$  \hspace{1cm} (7)

$$e^{-\frac{1}{2}(\nu+\lambda)} \frac{d}{dr} \left[ e^{\frac{1}{2}(\nu-\lambda)} \frac{dW}{dr} \right] - 2(1 + 2eW) \left[ \frac{W(1 + eW)}{r^2} + eQ^2 \right] = 0$$  \hspace{1cm} (8)

$$T^t_t + T^r_r = \frac{4}{r^2} \left[ (1 + 2eW)^2 Q^2 + \frac{2W^2(1 + eW)^2}{r^2} \right] + 2V(Q)$$

$$= -(8\pi G)^{-1} \left[ \frac{2(e^{-\lambda} - 1)}{r} + e^{-\lambda} \frac{d}{dr} (\nu - \lambda) \right],$$  \hspace{1cm} (9)

$$T^t_t + T^r_r = 2e^{-\lambda} \left[ \left( \frac{dQ}{dr} \right)^2 + \frac{2}{r^2} \left( \frac{dW}{dr} \right)^2 \right]$$

$$= (8\pi G)^{-1} e^{-\lambda} \frac{d}{dr} (\nu + \lambda).$$  \hspace{1cm} (10)

Far from the monopole core, the scalar field has the value $||\phi|| = \sqrt{2}v$, which corresponds to the true vacuum, that is $V(v) = 0 = dV/dQ(v)$, and the gauge field is $W = -(2e)^{-1} \equiv g$. In that region, the spacetime is described by the Reissner-Nordstrom metric of a magnetic monopole with charge $g$.

The norm of the timelike Killing vector is $e^\nu$. By definition, this norm vanishes if there is a black hole horizon. The other gravitational potential $\lambda$ must vanish at the origin in order to fulfill the condition of elementary flatness.

It is worth noticing that, according to Eq. (10), the derivative of $\lambda + \nu$ is always positive. Since $\lambda$ and $\nu$ vanish at infinity, it follows that $\lambda + \nu < 0$. For a black hole solution, $\nu \to -\infty$ before $\lambda \to \infty$, a fact which has been overlooked in Refs. [11] and [12], where only the potential $\lambda$ is evaluated. Notice also that, for a regular solution, $\lambda(0) = 0$ and $\nu(0) < 0$.

Near the center of the core, the fields have the approximate forms $Q = ar$ and $W = br^2$, which imply an equation of state

$$\epsilon \equiv T^t_t = (8\pi G)^{-1} \left[ 3(a^2 + 4b^2) + V(0) \right],$$  \hspace{1cm} (11)

$$p \equiv -T^r_r = (8\pi G)^{-1} \left[ a^2 - 4b^2 \right] + V(0),$$  \hspace{1cm} (12)
FIGURE 1. Dimensionless scalar field $q$ as a function of the dimensionless radial distance $x$, for three values of the parameter $\alpha'$, and for $\eta = 10^{-6}$.

FIGURE 2. Same as Fig. 1 for the dimensionless gauge field $w$.

Thus, the spacetime in the center of the monopole is not exactly DeSitter, as it is sometimes claimed.

Our integration method closely follows the one used in Ref. [3]. First, we assume that the potential $V$ has the form

$$V(Q) = \frac{1}{2} a(Q^2 - v^2)^2.$$  \hspace{1cm} (13)

Next, we set $Q = W = W' = \lambda = 0$ at $r = 0$, in order to guarantee that the solutions are regular (the potential $\nu$ can be eliminated from the equations). Finally, we choose particular values for $Q'$, $W''$ and $\lambda'$ at $r = 0$ and integrate from the origin to infinity, and look for solutions which tend asymptotically to $Q = v$, $W = -g$ and $e^{-\lambda} = 1 - 2GM/r + g^2/r^2$, where $M$ is the mass of the monopole which depends on the initial values for $Q'$, $W''$ and $\lambda'$.
3. RESULTS AND DISCUSSIONS

For the purpose of numerical calculations, it is convenient to define dimensionless functions $q$ and $w$ as $Q = vq, W = -w/2e$, and set $r = x/ev$. With this rescaling, the field equations contain only two parameters: $\alpha' \equiv \alpha/e^2$ and $\eta \equiv 8\pi G v^2$. The parameter $\eta$ is essentially the squared ratio of the GUT and Planck masses.

Values of $\eta$ in the interval $[10^{-8}, 10^{-1}]$ were chosen as representative. Within this range, all values of this parameter give essentially the same results up to a scaling factor of order $\eta$ for the potentials $\lambda$ and $\nu$. For larger values, black hole solutions exist according to the value of the parameter $\alpha'$, which is a measure of the strength of the scalar field potential; different values from 0 to 10 were taken for $\alpha'$.

In Figs. 1 and 2, we have plotted some typical solutions for the dimensionless functions
FIGURE 5. Critical value of $\eta(\eta^*)$ as a function of $\alpha'$.

$q$ and $w$, respectively. Due to the smallness of the gravitational effects, neither the scalar field $q$ nor the gauge field $w$ differ appreciably from the solutions found by Bogomol'nyi and Marinov [3].

In Figs. 3 and 4, we show the gravitational potentials which correspond to the functions in Figs. 1 and 2. Both $\lambda$ and $\nu$ remain finite and tend to the Reissner-Nordstrom solution when the dimensionless radial distance $x$ tends to infinity.

The typical values of the monopole mass turn out to be 1.234, 1.290 and 1.356 (in units of $v$) for $\alpha' = 0.3, 1$ and 3. These values are very close to those found by Bogomol'nyi and Marinov (1.160, 1.238 and 1.326 respectively) for the case without a gravitational field.

For $\eta$ above a critical value $\eta^*$, the function $\nu \to -\infty$, which implies that a black hole is formed. This critical value depends on the parameter $\alpha'$; the larger this last parameter, the smaller is the critical value of $\eta$ for the existence of a black hole solution. This result is plotted in Fig. 5, where we show the threshold value of $\eta$ for the disappearance of regular solutions and the existence of black hole horizons.

REFERENCES