Experimental analysis of chaos in underactuated electromechanical systems

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An underactuated system is a kind of non-autonomous robotic system in which there are more links than actuators. The complexity of the dynamical behavior of these systems allows a wide variety of steady-state responses. The reconstruction of attractors based on time series obtained from measurements of one of the variables of a two-link, planar, underactuated robot called Pendubot, is developed. Time-delay coordinates, average mutual information, and percentage of false nearest neighbors’ methods are used to reconstruct the invariant sets. It is shown that, under the action of a periodic torque, the Pendubot can display a variety of steady-state dynamics, including strange attractors.

Keywords: Chaos; time series analysis; robotics; attractor reconstruction

Un sistema electromecánico subactuado es un tipo de sistema robótico no autónomo que cuenta con más eslabones que actuadores. La complejidad del comportamiento dinámico de estos sistemas permite una gran variedad de respuestas en estado estacionario. En este trabajo se desarrolla la reconstrucción de atractores basada en series de tiempo obtenidas a partir de mediciones de una de las variables de un robot planar de dos grados de libertad subactuado llamado Pendubot. A fin de reconstruir los conjuntos invariantes, se utilizan técnicas como retraso de coordenadas, promedio de información mutua y porcentaje de falsos vecinos cercanos. Se muestra que bajo la acción de un torque periódico, el Pendubot puede desplegar una variedad de comportamientos dinámicos en estado estacionario incluyendo atractores extraños.

Descripciones: Caos; análisis de series de tiempo; robótica; reconstrucción de atractores

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1. Introduction

The dynamical analysis of certain systems is not always an easy task, particularly when the system dynamics involves terms that are difficult to know in a precise way, and the dynamics is seriously changed with small disturbances of these terms. These ones may include uncertain dynamics and parameters. Therefore, it is not always possible to have a precise mathematical model for the system we are dealing with. In this line of action there are lots of efforts which consider the use of measurements of, at least one of the variables of the system in order to characterize the dynamics of such systems, these set of techniques is known as time series analysis.

One interesting approach is the use of time-delayed versions of the time series from the measurements to reconstruct the corresponding limit set in the state space [1]. Although this powerful technique is suitable for systems showing simpler dynamics, it is particularly useful when the system dynamics exhibits complex behavior. In this case, obtaining fundamental invariants of the system, like the local dimension of the steady state dynamics and the reconstruction of the corresponding attractor is very important, but it is not an easy problem.

Chaotic dynamics have been widely studied in several disciplines during the last decades. A chaotic signal is generated by a deterministic dynamical system, but because of its sensitivity to initial conditions, it is long-term unpredictable. Some methods have been developed for situations where the system dynamics are known. However, when an accurate mathematical model is not available, time series analysis has shown to be a suitable alternative to this problem, this is the case for underactuated robots.

The important distinction between a standard robotic system and an underactuated robotic system is the absence of some actuator in the overall configuration of the device; this means that the underactuated system has more links than actuators and hence, some specific positions can not be reached via standard control strategies.

Underactuated manipulators arise in a number of important applications such as free-flying space robots, hyper-redundant manipulators and snake-like robots, manipulators
with structural flexibility, etc. Previous work on the modeling and control of such manipulators can be found in Refs. 2–4. Experimental analysis performed on different kinds of manipulators have shown the rich variety of steady states, ranging from equilibrium points to strange attractors.

In this paper we report the application of some reconstruction techniques in order to analyze the Pendubot’s [5] steady-state behavior when it is forced by a periodic torque. The Pendubot is a two links planar underactuated robot and the main objective of this paper is to analyze its complex dynamics in order to help in the further selection of a suitable control strategy. This analysis relies only on the availability of a time series, obtained from a measured variable of the system. Therefore, given a time series obtained from some measured variables of the Pendubot, possibly showing an irregular behavior, the objective is to determine the stochastic or deterministic nature of the system dynamics, as well as some fundamental parameters. In particular, we are interested in calculating the local dimension of the system dynamics, and to reconstruct the corresponding attractor.

The paper is organized as follows. In Sec. 2 some basic concepts on dynamical systems are reviewed. Section 3 gives a description of some methods to analyze signals arising from chaotic systems. In Sec. 4 an application of these methods to the Pendubot, and some experimental results, are shown. Finally, Sec. 5 includes some conclusions.

2. Some concepts on dynamical systems

Consider a system given by

$$\dot{x} = f(x, t, \mu),$$

(1)

where $x \in \mathbb{R}^n$ is the state, $f: \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, and $\mu$ denotes the system parameters. The solution of Eq. (1) is some vector function $x = x(t)$ that describes the trajectories in the state space constructed with its coordinates. Depending on the parameter values the system may display different steady states, ranging from equilibrium points to chaotic attractors.

**Definition 1. (Chaotic Attractor) [6]**. Consider an autonomous vector field $C^r$ ($r \geq 1$) on $\mathbb{R}^n$, defining a system like (1). Denote the flow generated by (1) as $\phi(t, x)$, and assume that $\Lambda \subset \mathbb{R}^n$ is a compact set, invariant under $\phi(t, x)$. Then $\Lambda$ is said to be chaotic if show the following behavior:

1) **Sensitive dependence on initial conditions.** There exist $\varepsilon > 0$ such that for any $x \in \Lambda$ and any neighborhood $U$ of $x$, there exist $y \in U$ and $t > 0$ such that $|\phi(t, x) - \phi(t, y)| > \varepsilon$.

2) **Topological transitivity.** For any two open sets $U$, $V \subset \Lambda$, there exists $t \in \mathbb{R}$ such that $\phi(t, U) \cap V \neq \emptyset$.

They are not easy to analyze due to the absence of tools allowing a good understanding of the phenomena. In recent years, many techniques have been developed for the analysis of the dynamics of this kind of systems. Some of these techniques are described in the next section, but before we will give some definitions and a useful theorem.

**Definition 2. (Capacity Dimension) [7]**. Let $A$ be a bounded subset of $\mathbb{R}^n$. Let $N_\delta(A)$ be the smallest number of sets of maximum diameter $\delta$ that cover $A$. Then, the capacity dimension is defined, if exists, by

$$\dim_K(A) = \lim_{\delta \to 0} \frac{\log N_\delta(A)}{\log(1/\delta)},$$

(2)

Typically, this quantity is not an integer number for a chaotic attractor $A$. When this situation occurs it is said that $A$ is a fractal set.

**Theorem. (Embedding Theorem) [1]**. Let $A$ be a compact and $E$ a subspace of finite dimension such that

$$\dim_E > 2 \dim_K(A) + 1,$$

(3)

where $\dim_K$ is the capacity dimension. Then the set of projections $\pi: A \to E$, such that $\pi$ is injective, is dense among all projections with respect to the norm operator topology.

**Definition 3. (Embedding Dimension).** The dimension $\dim_E = \dim_K$ in Eq. (3) is called the embedding dimension and it is the dimension for which the attractor is fully unfolded; i.e., the dimension in which two points far away the other in the original space are not projected near each other in the observation space.

Due to this theorem, it is possible to reconstruct the attractor in some previously determined embedding dimension. The problem here is to find this dimension from a time series. In the next sections some prescriptions for finding this dimension, and some other necessary parameters for the attractor reconstruction will be given.

3. Analysis of chaotic systems

There are no analytical solutions to equations describing chaotic phenomena, even an approximate solution is not easy to find. Some analysis techniques for this kind of systems involve perturbation methods [7] for setting approximate solutions of Eq. (1). An important point here is that usually, it is possible to measure at least one of the variables involved in the time evolution of the system. There are some methods to analyze the chaotic phenomena by using time series. These methods are based on the embedding theorem for the reconstruction of the attractor, and some prescriptions have been proposed to calculate some important system parameters [9–11]. Due to this theorem, it is possible to reconstruct the attractor if the embedding dimension is previously determined. Two problems arise here, the first one is how to find this dimension from a time series and the second one is how to determine the time delay. In what follows some prescriptions for solving this problems, and some other necessary parameters for the attractor reconstruction, will be given.
3.1. Attractor reconstruction

There are several procedures to reconstruct a chaotic attractor from discrete time measurements [1, 7, 11]. In general, the solution relies on choosing a suitable sampling period for the signal such that topological characteristics of the attractor can be reproduced. The attractor reconstruction is then accomplished by using time delay versions of a scalar quantity $s(t)$, observed from time $t_0$ to some final time, as coordinates for the state space. Let us define $x(n) = s(t_0 + n\Delta t)$, $n = 1, 2, \ldots$, for some initial time $t_0$ and a sampling interval $\Delta t$. From the observations, $d$-dimensional vectors

$$ y(n) = \{ x(n), x(n + T), \ldots, x[n + (d - 1)T] \} \quad (4) $$

are used to trace out the orbit of the system. Thus, the problem arising here is what values of time delay factor $T$ and the embedding dimension $d = d_E$ to choose. The next two subsections deal with those problems.

3.2. Average mutual information (AMI)

Before formally describing the idea of mutual information, we have to consider some restrictions. First, if the period $T$ is too short, coordinates $x(n)$ and $x(n + T)$ would not be independent enough. And second, if $T$ is too large, every connection between these coordinates would be numerically subject to be random-like with respect to each other. In Ref. 10 it is suggested to base the selection of $T$ in a fundamental aspect of chaos: the information generation. The average mutual information concept is based on the Shannon's idea for information. Let us consider two measurements $a_i$ and $b_j$ from sets $A = a_i = x(n)$ and $B = b_j = x(n + T)$ respectively. The mutual information between measurement $a_i$ and measurement $b_j$ is the quantity learned by measurement $a_i$ about measurement $b_j$. In bits, this is given as follows:

$$ I(T) = \sum_{x(n), x(n + T)} P[x(n), x(n + T)] \log_2 \left[ \frac{P[x(n), x(n + T)]}{P[x(n)]P[x(n + T)]} \right]. \quad (5) $$

The prescription for determining if the values of $x(n)$ and $x(n + T)$ are independent is such that we can use them to construct the vector $y(n)$, is to take $T$ where the first minimum of the $I(T)$ occurs.

3.3. Global false nearest neighbors

Theorem tells us that if the attractor dimension defined by the orbits associated to Eq. (1) is $d_E = \dim_K(A)$, then the attractor will unfold in an integer embedding dimension $d_E > 2 \dim_K(A) + 1$ as a maximum value. In an embedding dimension that is too small to unfold the attractor, not all the points that lie close to each other will be neighbors because of the dynamics, some of them will actually be far from each other and appear as neighbors, because the geometric structure of the attractor has been projected down onto a smaller space. In a $d$-dimensional space and denoting the $r$-th nearest neighbor of $y(n)$ by $y_r(n)$, the square of the euclidean distance between these two points is given by [9]

$$ R^2_d(n,r) = \sum_{k=0}^{d-1} [x(n + kT) - x_r(n + kT)]^2. \quad (8) $$

A criterion to find false neighbors may be the increase in distance between $y(n)$ and $y_r(n)$ when going from dimension $d$ to $d + 1$. The increase of distance can be stated as [7]:

$$ \sqrt{\frac{R^2_{d+1}(n,r) - R^2_d(n,r)}{R^2_d(n,r)}} > R_{TH}, \quad (9) $$

where $R_{TH}$ is some threshold number. In our case $R_{TH} \geq 15$, was founded experimentally.

4. Application to the Pendubot

We have chosen an underactuated mechatronic system that presents a wide variety of behaviors. The system called Pendubot [5] consists of two rigid links, link 1 is directly coupled to the shaft of a 90 V permanent magnet DC motor mounted to the end of a table, this motor is the only one actuator of the system. Link 2 is coupled to link 1 and is moved only by the motion of link 1. The angular position of both links is monitored to a computer via optical encoders as shown in Fig. 1.
All of our computations were performed on a personal computer with a D/A card and an encoder interface card. Using the standard software library routines supplied with these interface cards we were able to program our algorithms directly in C-language. The voltage to the DC motor is supplied via a servo amplifier. There is a relationship between the supplied voltage and the applied torque to the DC motor, this relationship was experimentally determined to be

\[ T = (l_1 w_2 + l_{e1} w_2) \cos \left( \frac{\pi}{2} - (0.2763 V + 0.0335) \right), \]  

where \( T \) is the applied torque and \( V \) is the voltage supplied to the power amplifier. The parameters used here were measured directly from the device and are: \( l_1 = 0.26987 \text{ m}, \, w_1 = 5.1885 \text{ N}, \, l_{e1} = 0.13494 \text{ m}, \) and \( w_2 = 3.2824 \text{ N} \).

We have applied several sinusoidal voltage inputs as \( V = A \sin \omega t \) V to the Pendubot and measured the angular position in the second link (the one without an actuator). The frequency used was \( \omega = 4 \text{ rad/s} \) and we have varied the voltage amplitude from 0.1 V to 1.7 V. In the following, we show some selected examples of attractor reconstruction applying the average mutual information criterion for finding a suitable \( T \) and the false neighbors idea to find the embedding dimension.

The angular position of the second link of the Pendubot has been sampled every 16 ms and we have taken the transients off the time series.

**Example 1. Periodic Orbit.** We have considered an amplitude \( A = 0.5 \text{ V} \). Figure 2a shows the AMI for this signal. Using the criteria previously suggested we have taken \( T = 13 \). Figure 2b shows the percentage of false nearest neighbors for this value of \( T \). Note that once the percentage of false nearest neighbors has reached the zero value, this percentage does not change anymore. The first value of \( d \) where the percentage is zero is the embedding dimension \( d_E \).

Figure 3 displays the reconstructed attractor using \( T = 13 \) and \( d_E = 3 \). It can be observed from this Figure that the reconstructed attractor is a periodic limit cycle, indeed this is a period-3 stable orbit. Thus we know that there should be some set of parameters \( \mu \) for which the system would exhibit chaotic behavior [12].
Example 2. Chaotic attractor. We have analyzed the dynamics for an input signal with an amplitude $A = 0.9$ V. Figure 4a shows the average mutual information as a function of the time delay $T$, applying the prescription previously described, we find the embedding dimension to be $d_E = 3$, this is shown in Fig. 4b. In Fig. 5 it can be observed the reconstructed attractor.

In order to prove that this behavior is chaotic, we have obtained the largest Lyapunov exponent (LE) [13] resulting $\lambda_1 = 0.092$. This positive $\lambda_1$ indicates that the time series is chaotic, although positive LE's also describe noisy signals we can say that the behavior of the system for $A = 0.9$ V is chaotic because: a) the AMI of the signal has a strict local minimum, b) the percentage of false nearest neighbors falls to zero in some finite dimension, c) the geometry of the reconstructed attractor, and d) the signal has a broad frequency spectrum. The frequency spectrum of the measurements was obtained using a standard FFT algorithm and is shown in Fig. 6.

Example 3. Chaotic Attractor. The input signal considered now has an amplitude $A = 1.4$ V. Figure 7a shows the average mutual information as a function of the time delay $T$. In Fig. 7b it is shown the percentage of false neighbors giving $d_E = 3$. Figure 8 shows the attractor reconstruction for these parameters.

5. Conclusions

The main objective for using a manipulator robot is to accomplish tasks involving the exact tracking of some desired trajectory. The exact tracking depends on the nature of the
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designed control algorithm, provided the proper analysis of the dynamics of the particular device. The absence of an actuator may lead to an improper performance of the system, bad tracking of the desired trajectory and also to a loss of stability. In addition, as we have shown, underactuated robots like the Pendubot may exhibit chaotic behavior for a certain set of parameters.

There is not a general framework for chaos control and suppressing undesired complex dynamics, but it is needed, as a first stage, the analysis of the dynamics of the system to be controlled. In this paper we have shown a way to analyze the complex behavior of the Pendubot on the basis of having measurements of only one of the variables of the system. After performing several experiments we have noticed about the existence of chaotic responses in this system. These methods help us in analyzing the dynamics of any system requiring only a measurement of one of its variables, particularly they are useful for extracting information from the system and besides, they can be the first stage in the application of the method developed by Ott, Grebogi, and Yorke [14] for controlling chaotic behavior.

The average mutual information (AMI) is useful because it suggests a suitable sampling period for the signal and it offers a first sight about the stochastic or deterministic nature of the signal because if we are dealing with a noisy signal the AMI does not have a strict local minimum. Finding the global false nearest neighbors helps us in determining the dimension of the space where the attractor can be observed; i.e. where the essential dynamics in stationary state is embedded; as the AMI, this method is useful for determining the nature of the signal, because noisy signals are typically embedded in an infinite-dimensional space.

Finally, we can say that the attractor reconstruction displays the geometry of an object topologically equivalent to the attractor constructed with the original coordinates. Besides, it offers a clear view of the different kinds of behavior displayed by the system. In this sense, it complements the information provided by the spectrum and the LE of the signal.