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Electromagnetic quasinormal modes of five-dimensional topological black holes

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We calculate exactly the QNF of the vector type and scalar type electromagnetic fields propagating on a family of five-dimensional topological black holes. To get a discrete spectrum of quasinormal frequencies for the scalar type electromagnetic field we find that it is necessary to change the boundary condition usually imposed at the asymptotic region. Furthermore for the vector type electromagnetic field we impose the usual boundary condition at the asymptotic region and we discuss the existence of unstable quasinormal modes in the five-dimensional topological black holes.

Keywords: Quasinormal modes; electromagnetic field; five-dimensional topological black holes.

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1. Introduction

The quasinormal modes (QNM) are the characteristic oscillations of a test field or of the metric perturbations that satisfy the appropriate boundary conditions near the event horizon and at the asymptotic region of the black hole [1,2]. Thus the QNM appear when the black hole is perturbed from its equilibrium state and they depend on the physical properties of the black hole and the field. Recently the QNM have found many applications in different studies about the classical stability of the black holes [1,2], the area spectrum of the event horizon [3,4], and motivated by the AdS-CFT correspondence [5], the QNM spectra of asymptotically anti-de Sitter black holes are extensively studied since they are useful in the calculation of the decay rates in dual theories [5-24].

Furthermore the exactly solvable systems are relevant in physics, since in these systems we can explore in detail their physical properties. Thus we think that in the research line of black hole perturbations is useful the search and study of exactly solvable systems. In particular for asymptotically anti-de Sitter black holes it is possible to calculate exactly the quasinormal frequencies (QNF) of several fields [18-25]. We know that this is the case for the BTZ black holes [18-20], the massless topological black holes [21-24], and the five-dimensional topological black holes [25] of Refs. 26 to 28. See also [29-31] for related examples of exact determination of QNF.

The QNF spectrum of the electromagnetic field in asymptotically anti-de Sitter black holes has been explored previously [8-12], [14-17]. Among the motivations we find that the electromagnetic field behaves in a different way than other classical fields and its analysis is physically more relevant than the study of the Klein-Gordon field. Additionally in the AdS-CFT correspondence the QNF of the electromagnetic field in asymptotically anti-de Sitter spacetimes are related to the poles of the retarded Green functions of the R-symmetry currents. Thus for the electromagnetic field we believe that it is convenient to determine its spectrum of QNF in asymptotically anti-de Sitter black holes.

Based on the results by Kodama and Ishibashi on the simplification of the Maxwell equations in static spacetimes [32], (see also [33]) here we generalize previous results on the QNM of the Klein-Gordon field [25] in the five-dimensional topological black holes of Refs. 26 to 28 and calculate exactly the spectrum of QNF for the electromagnetic field. Using these results we study the stability of the topological black holes of Refs. 26 to 28 under perturbations. At this point it is convenient to mention that we discuss the possible existence of unstable QNM for the electromagnetic field for some values of the parameters.

We organize this work as follows. Following Kodama and Ishibashi [32] (see also [33]) in Sec. 2 we recall some relevant results about the simplification of the vacuum Maxwell equations in static spacetimes to two differential equations, one for the vector type electromagnetic field and another for the scalar type electromagnetic field. In Sec. 3 we enumerate the relevant features of the five-dimensional topological black holes of Refs. 26 to 28 that we study in this work. In Sec. 4 we calculate exactly the QNF of the vector type electromagnetic field propagating on these five-dimensional topological black holes and discuss the stability of the QNM. We make a similar calculation for the scalar type electromagnetic field in Sec. 5, but in this case we need to make a careful study of the boundary condition at the asymptotic region since the usual boundary condition leads to a continuum of QNF and we modify the boundary condition to get a discrete set of QNF that depends on the parameters of the black hole and the field. Finally we discuss the main results in Sec. 6.

2. Maxwell equations

As is well known, we can write the line element of a D-dimensional generalization of the spherically symmetric spacetime in the form [32]

$$\mathrm{d}s^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b + r^2(x)\mathrm{d}\Omega^2_{D-2}, \tag{1}$$

where $a, b = 1, 2$, $\mathrm{d}\Omega^2_{D-2} = \hat{\gamma}_{ij}\mathrm{d}\hat{y}^i\mathrm{d}\hat{y}^j$, $i, j = 1, 2, \ldots, D-2$, is the line element of the $(D-2)$-dimensional maximally symmetric base manifold with metric $\hat{\gamma}_{ij}$ and whose Ricci tensor fulfills $\hat{R}_{ij} = (D-3)K\hat{\gamma}_{ij}$, that is, the base manifolds are of Einstein type. Here K is a constant determined by the scalar curvature of the base manifold and can be normalized to the values $K = 0, \pm 1$ [32]. In what follows we assume that the bidimensional line element that appears in the metric (1) is given by

$$\mathrm{d}s_2^2 = g_{ab}(x)\mathrm{d}x^a\mathrm{d}x^b = -\mathcal{F}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathcal{G}}, \tag{2}$$

with $\mathcal{F}$ and $\mathcal{G}$ functions of the radial coordinate r.

If we denote the Maxwell tensor by $F_{\mu\nu}$ then the Maxwell equations in vacuum are

$$\nabla_{[\sigma}F_{\mu\nu]} = 0, \qquad \nabla_\mu F^{\mu\nu} = 0. \tag{3}$$

It is well known that if we make a harmonic sum on the scalar and vector eigenfunctions of the Laplacian on the base manifold $\mathrm{d}\Omega^2_{D-2}$, the Maxwell equations in a spacetime of the form (1) simplify to [32,33].

$$D_a D^a \Phi_V - \frac{D-4}{4r}\frac{\mathrm{d}\mathcal{G}}{\mathrm{d}r}\Phi_V - \frac{(D-4)(D-6)\mathcal{G}}{4r^2}\Phi_V - \frac{D-4}{4r}\frac{\mathcal{G}}{\mathcal{F}}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}r}\Phi_V - \frac{k_V^2 + (D-3)K}{r^2}\Phi_V = 0, \tag{4}$$

for the vector type electromagnetic field and

$$D_a D^a \Phi_S - \frac{(D-2)(D-4)}{4}\frac{\mathcal{G}}{r^2}\Phi_S + \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}r}\frac{D-4}{4r}\Phi_S + \frac{\mathcal{G}}{r\mathcal{F}}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}r}\frac{D-4}{4}\Phi_S - \frac{k_S^2}{r^2}\Phi_S = 0, \tag{5}$$

for the scalar type electromagnetic field. Here the symbol D_a denotes the covariant derivative for the bidimensional metric $g_{ab}(x)$, the functions Φ_V and Φ_S depend on the coordinates x^a of the two-dimensional space with metric g_{ab} and they contain the relevant information about the dynamics of the vector type and scalar type electromagnetic fields in spacetimes of the form (1). In the previous formulas k_V^2 (k_S^2) are the eigenvalues of the vector harmonics $\mathbb{V}_i$ (scalar harmonics $\mathbb{S}$) on the maximally symmetric base manifold with line element $\mathrm{d}\Omega^2_{D-2}$, that is, they satisfy [32]

$$(\hat{D}_i\hat{D}^i + k_V^2)\mathbb{V}_j = 0, \qquad \hat{D}^i\mathbb{V}_i = 0, \qquad ((\hat{D}_i\hat{D}^i + k_S^2)\mathbb{S} = 0), \tag{6}$$

where $\hat{D}_i$ is the covariant derivative on the maximally symmetric base manifold. For $\mathcal{F} = \mathcal{G} = f$ we point out that in Eqs. (4) and (5) the operator $D_a D^a$ takes the form

$$D_a D^a = -\frac{1}{f}\partial_t^2 + \partial_r(f\partial_r). \tag{7}$$

3. Five dimensional topological black holes

The five-dimensional topological black holes that we study in this work have the line element

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f} + r^2\mathrm{d}\Omega_K^2, \tag{8}$$

where $\mathrm{d}\Omega_K^2$ is the line element of the three-dimensional maximally symmetric base manifold and the function f takes the form [26–28]

$$f = -\frac{\Lambda}{3}r^2 + K \pm \sqrt{c_0}, \tag{9}$$

where Λ is a negative constant, $K = 0, \pm 1$, and c_0 is a non negative constant. We notice that for the three-dimensional base manifold the scalar curvature is equal to $6K$. The solution with positive sign of $\sqrt{c_0}$ is usually called the plus branch, whereas the solution with negative sign of $\sqrt{c_0}$ is usually known as the minus branch [28]. The topological black holes (8) are solutions of several gravity theories as the five-dimensional Chern-Simmons theory [26], the five-dimensional Gauss-Bonnet gravity with special Gauss-Bonnet coefficient [27], and the five-dimensional $z = 4$ Hořava-Lifshitz gravity [28].

The event horizon of the topological black holes (8) is determined by

$$r_+^2 = -\frac{3}{\Lambda}(\mp\sqrt{c_0} - K). \tag{10}$$

Since we impose that the radius of the event horizon is positive, we need that the parameters c_0 and K fulfill

$$\mp\sqrt{c_0} - K > 0. \tag{11}$$

Notice that depending on the value of K we have one or two positive values of the event horizon according to the following list.

1. For $K = 0$ and $K = 1$ the black hole exists in the minus branch.
2. For $K = -1$ to have a black hole in the plus branch we require that $\sqrt{c_0} < 1$.
3. For $K = -1$ in the minus branch we always have a black hole.

Thus for $K = -1$ we have black holes in the two branches of the solution (8) [28].

In the following sections it is useful to define the quantity

$$p = K \pm \sqrt{c_0}, \tag{12}$$

and if the black hole exists, then $p < 0$. We also notice that for $K = -1$ the parameter p satisfies $|p| > 1$ in the minus branch and $|p| < 1$ in the plus branch.

4. Quasinormal frequencies of the vector type electromagnetic field

To extend the previous results of [25] on the QNM spectrum of the five-dimensional topological black holes (8) in what follows we calculate exactly the QNF of the electromagnetic field moving on these black holes. Following Refs. 18 to 25 for the topological black holes (8) we define their QNM as the oscillations that satisfy the boundary conditions

$i)$ The electromagnetic field is purely ingoing near the black hole horizon.

$ii)$ The electromagnetic field goes to zero as $r \to +\infty$.

We notice that the line elements of the five-dimensional topological black holes (8) are of the form (1) with $\mathcal{F}=\mathcal{G}=f$, therefore we can use Eqs. (4) and (5) to study the propagation of the electromagnetic fields on these backgrounds. Here we begin with the vector type electromagnetic field.

Since the topological black holes (8) are static and taking into account the expression (7) for the operator $D_a D^a$, we propose that the function Φ_V takes the form

$$\Phi_V = \mathrm{e}^{-i\omega t} R_V(r), \tag{13}$$

and therefore from Eq. (4) we obtain that the radial function R_V must be a solution of the differential equation

$$f \frac{\mathrm{d}^2 R_V}{\mathrm{d}r^2} + \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}R_V}{\mathrm{d}r} + \left(\frac{\omega^2}{f} - \frac{1}{2r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{f}{4r^2} - \frac{k_V^2 + 2K}{r^2}\right) R_V = 0. \tag{14}$$

To solve exactly the previous differential equation, as in Ref. 25, it is convenient to define the variable[i]

$$v = 1 - \frac{3p}{\Lambda r^2}, \tag{15}$$

with p already given in the formula (12). Using the variable v we find that Eq. (14) transforms into

$$\frac{\mathrm{d}^2 R_V}{\mathrm{d}v^2} + \left(\frac{1}{v} - \frac{1/2}{1-v}\right)\frac{\mathrm{d}R_V}{\mathrm{d}v} + \left(\frac{F+G-3/16}{v(1-v)} + \frac{F}{v^2} - \frac{3/16}{(1-v)^2}\right) R_V = 0, \tag{16}$$

where we define the constants F and G by

$$F = \frac{3\omega^2}{4p\Lambda}, \qquad G = \frac{k_V^2 + 2K}{4p} - \frac{1}{16}. \tag{17}$$

Taking the function R_V in the form

$$R_V = (1-v)^{A_V} v^{B_V} R_{2V}, \tag{18}$$

with the parameters A_V and B_V being solutions to the algebraic equations

$$A_V^2 - \frac{A_V}{2} - \frac{3}{16} = 0, \qquad B_V^2 + F = 0, \tag{19}$$

we find that the function R_{2V} is a solution of the differential equation

$$v(1-v)\frac{\mathrm{d}^2 R_{2V}}{\mathrm{d}v^2} + (2B_V + 1 - (2B_V + 2A_V + 3/2)v)\frac{\mathrm{d}R_{2V}}{\mathrm{d}v} - (2A_V B_V + B_V/2 + A_V + 3/16 - F - G)R_{2V} = 0. \tag{20}$$

This equation is of hypergeometric type [34-36]

$$v(1-v)\frac{\mathrm{d}^2 h}{\mathrm{d}v^2} + (\gamma - (\alpha+\beta+1)v)\frac{\mathrm{d}h}{\mathrm{d}v} - \alpha\beta h = 0, \tag{21}$$

with parameters α_V, β_V, γ_V equal to

$$\alpha_V = A_V + B_V + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{\tfrac{1}{4} + 4G},$$
$$\beta_V = A_V + B_V + \tfrac{1}{4} - \tfrac{1}{2}\sqrt{\tfrac{1}{4} + 4G},$$
$$\gamma_V = 2B_V + 1. \tag{22}$$

In what follows we choose

$$A_V = \frac{3}{4}, \qquad B_V = \frac{i\omega q}{2}, \tag{23}$$

where

$$q = \sqrt{\frac{3}{p\Lambda}}. \tag{24}$$

Notice that $q > 0$. For these values of A_V, B_V we get that $\gamma_V - \alpha_V - \beta_V = -1$.

From these results we find that the function R_V is equal to [34-36]

$$R_V = (1-v)^{3/4} v^{i\omega q/2}\big(K_1\, {}_2F_1(\alpha_V, \beta_V; \gamma_V; v) + K_2 v^{1-\gamma_V}\, {}_2F_1(\alpha_V - \gamma_V + 1, \beta_V - \gamma_V + 1; 2 - \gamma_V; v)\big), \tag{25}$$

where K_1, K_2 are constants and ${}_2F_1(\alpha, \beta; \gamma; v)$ denotes the hypergeometric function [34]– [36]. Taking into account that $v \approx 0$ near the horizon of the black hole, from the expression (25) for R_V, we observe that near the horizon this function behaves as

$$R_V \approx K_1 v^{i\omega q/2} + K_2 v^{-i\omega q/2} \approx K_1 \mathrm{e}^{i\omega r_*} + K_2 \mathrm{e}^{-i\omega r_*}, \tag{26}$$

where r_* denotes the tortoise coordinate of the five-dimensional topological black hole (8)

$$r_* = \frac{q}{2}\ln\left|\frac{\sqrt{1-v}-1}{\sqrt{1-v}+1}\right|. \tag{27}$$

Notice that $r_* \to -\infty$ near the horizon and $r_* \to 0$ as $r \to +\infty$.

Since we take a time dependence of the form $\exp(-i\omega t)$ (see the expression (13)) we find that the first term of the expression (26) is an outgoing wave near the horizon, whereas the second term is an ingoing wave near the horizon. Thus to satisfy the boundary condition i) of the QNM we must take $K_1 = 0$ and therefore the function R_V fulfilling the boundary condition near the horizon takes the form

$$\begin{aligned} R_V &= K_2(1-v)^{3/4} v^{-i\omega q/2} {}_2F_1(\alpha_V - \gamma_V + 1, \\ &\quad \beta_V - \gamma_V + 1; 2 - \gamma_V; v) \\ &= K_2(1-v)^{3/4} v^{-i\omega q/2} {}_2F_1(\hat{\alpha}_V, \hat{\beta}_V; \hat{\gamma}_V; v), \end{aligned} \tag{28}$$

with

$$\begin{aligned} \hat{\alpha}_V &= \alpha_V - \gamma_V + 1, \qquad \hat{\beta}_V = \beta_V - \gamma_V + 1, \\ \hat{\gamma}_V &= 2 - \gamma_V. \end{aligned} \tag{29}$$

To study the behavior of the field as $r \to +\infty$ ($v \to 1$) the usual procedure for many exactly solvable problems [18-25] is to use the Kummer property of the hypergeometric function [34-36], but for the vector type field, since the parameters of the hypergeometric function that appears in the solution (28) fulfill $\hat{\gamma}_V - \hat{\beta}_V - \hat{\alpha}_V = -1$ and the Kummer property is not valid when the values of the parameters satisfy this condition [34-36], we cannot employ the usual procedure. Nevertheless for $\gamma - \alpha - \beta = -m$, $m = 0, 1, 2, \ldots$, the hypergeometric function satisfies

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;v) &= \frac{\Gamma(\gamma)\Gamma(m)}{\Gamma(\alpha)\Gamma(\beta)}(1-v)^{-m} \\ &\quad \times \sum_{s=0}^{m-1} \frac{(\alpha-m)_s(\beta-m)_s}{s!(1-m)_s}(1-v)^s \\ &\quad + \frac{(-1)^{m+1}\Gamma(\gamma)}{\Gamma(\alpha-m)\Gamma(\beta-m)} \\ &\quad \times \sum_{s=0}^{\infty} \frac{(\alpha)_s(\beta)_s}{s!(m+s)!}(1-v)^s \\ &\quad \times [\ln(1-v) - \psi(s+1) \\ &\quad - \psi(s+m+1) + \psi(\alpha+s) + \psi(\beta+s)], \end{aligned} \tag{30}$$

when the parameters α and β are different from negative integers [34-36]. In the previous formula ψ is the logarithmic derivative of the gamma function and $(m)_s$ is the Pochhammer symbol. For $m = 0$ we must delete the finite sum. Notice that for the vector type electromagnetic field we have $m = 1$.

To analyze the behavior of the field as $r \to +\infty$ we use the property (30) to write the radial function (28) as

$$\begin{aligned} R_V &= K_2(1-v)^{3/4} v^{-i\omega q/2} \Bigg(\frac{\Gamma(\hat{\gamma}_V)}{\Gamma(\hat{\alpha}_V)\Gamma(\hat{\beta}_V)} \frac{1}{1-v} \\ &\quad + \frac{\Gamma(\hat{\gamma}_V)}{\Gamma(\hat{\alpha}_V - 1)\Gamma(\hat{\beta}_V - 1)} \sum_{s=0}^{\infty} \frac{(\hat{\alpha}_V)_s(\hat{\beta}_V)_s}{s!(s+1)!}(1-v)^s \\ &\quad \times \big[\ln(1-v) - \psi(s+1) - \psi(s+2) \\ &\quad + \psi(\hat{\alpha}_V + s) + \psi(\hat{\beta}_V + s) \big] \Bigg). \end{aligned} \tag{31}$$

From this expression we obtain that the second factor goes to zero as $r \to +\infty$, but in this limit the first factor behaves in the form

$$\frac{\Gamma(\hat{\gamma}_V)}{\Gamma(\hat{\alpha}_V)\Gamma(\hat{\beta}_V)} \frac{1}{(1-v)^{1/4}}, \tag{32}$$

and therefore diverges as $v \to 1$. In a similar way to previous works [18-25,33], to cancel this term we like to impose the conditions

$$\hat{\alpha}_V = -n, \qquad \hat{\beta}_V = -n, \tag{33}$$

with $n = 0, 1, 2, \ldots$ But if we impose the conditions (33), then we contradict the assumptions under which the property (30) is true. Thus if we use the property (30) then we cannot impose the conditions (33) because we contradict the assumptions under which this property is valid.

Another path is to impose the conditions (33) on the parameters of the solution (28) and see whether the radial function R_V satisfies the boundary condition ii) of the QNM [33]. Thus assuming that $\hat{\alpha}_V = -n$, we find that

$$\begin{aligned} R_V &= K_2(1-v)^{3/4} v^{-i\omega q/2} {}_2F_1(-n, \hat{\beta}_V; \hat{\gamma}_V; v) \\ &= K_2(1-v)^{3/4} v^{-i\omega q/2} \frac{(1-\beta_V)_n}{(2-\gamma_V)_n} \\ &\quad \times {}_2F_1(-n, \beta_V - \gamma_V + 1; \beta_V - n; 1-v), \end{aligned} \tag{34}$$

where we use that the hypergeometric function fulfills [36]

$$\begin{aligned} {}_2F_1(-n,\beta;\gamma;v) &= \frac{(\gamma-\beta)_n}{(\gamma)_n} \\ &\quad \times {}_2F_1(-n, \beta; \beta - \gamma - n + 1; 1-v). \end{aligned} \tag{35}$$

In a straightforward way we obtain that near $v = 1$ the function R_V of the expression (34) behaves as

$$R_V \approx (1-v)^{3/4}, \tag{36}$$

and therefore it satisfies the boundary condition ii) of the QNM as $v \to 1$. A similar thing happens for $\hat{\beta}_V = -n$. Thus imposing the conditions (33) we obtain that the radial function R_V satisfies the boundary condition ii) of the QNM.

From the conditions (33) and taking into account the values of the parameters (22) and (23) we get that the QNF of the vector type electromagnetic field are equal to

$$\omega_V = \pm \frac{1}{q} \sqrt{\frac{k_V^2 + 2K}{|p|}} - i\frac{2}{q}(n+1). \tag{37}$$

It is convenient to notice that the QNF (37) of the vector type electromagnetic field depend on the value of the parameter K related to the scalar curvature of the base manifold. We recall that $p < 0$ to have a positive radius of the event horizon. Furthermore we remember that the eigenvalues k_V^2 are non negative and they are discrete for $K = 1$, whereas they are continuous for $K \leq 0$ [32]. From the values (37) for the QNF we get that for $K = 0$ and $K = 1$ the real part of the QNF for the vector type field is different from zero and the imaginary part is negative, that is, the QNM decay in time since we have a time dependence $\exp(-i\omega t)$, and therefore they are stable for $K = 0$ and $K = 1$. We remark that for $K = 0$ and $K = 1$ the QNF (37) are complex.

For $K = -1$ we obtain that the QNF (37) simplify to

$$\omega_V = \pm \frac{1}{q}\sqrt{\frac{k_V^2 - 2}{|p|}} - i\frac{2}{q}(n+1). \tag{38}$$

Thus for $k_V^2 > 2$ the QNF (38) have real and imaginary parts different from zero and they are stable since $\mathbb{Im}(\omega_V) < 0$. Nevertheless if the base manifold allows eigenvalues of the vector harmonics satisfying $0 < k_V^2 < 2$, we obtain that the QNF (38) transform into

$$\omega_V = \pm \frac{i}{q}\sqrt{\left|\frac{k_V^2 - 2}{p}\right|} - i\frac{2}{q}(n+1), \tag{39}$$

that is, they are purely imaginary. From the expressions (38) and (39) we remark that for $K = -1$ we get complex or purely imaginary QNF depending on the value of the eigenvalues k_V^2. The QNF (39) with the minus sign of the square root are stable since $\mathbb{Im}(\omega_V) < 0$, but the QNF with the plus sign of the square root are unstable if

$$\left|\frac{k_V^2 - 2}{p}\right| > 4(n+1)^2. \tag{40}$$

We notice that this inequality is not satisfied for $|p| > 1$, but for k_V^2 sufficiently small, we can fulfill the inequality (40) for sufficiently small $|p| < 1$, that is, for the plus branch of the topological black holes (8) with $K = -1$. Thus, for these topological black holes the fundamental mode ($n = 0$) is unstable for

$$2 - 4|p| > k_V^2. \tag{41}$$

For the overtones we find that the condition (40) becomes

$$2 - 4|p|(n+1)^2 > k_V^2. \tag{42}$$

Thus for the topological black holes (8) with $K = -1$ in the plus branch with sufficiently small values of the parameter $|p|$ we find unstable QNM for the vector type electromagnetic field if the base manifold has eigenvalues of the vector harmonics in the range $0 < k_V^2 < 2$.

Nevertheless, for three-dimensional closed base manifolds and three-dimensional open base manifolds such that the quantity $\mathbb{V}^j \hat{D}_i \mathbb{V}_j$ fall off sufficiently rapidly at infinity, it is known that the eigenvalues of the vector harmonics satisfy $k_V^2 \geq 2|K|$ [32]. Thus for these three-dimensional base manifolds there is no instability of the QNM of the vector type electromagnetic field. We do not know an example of a three-dimensional base Einstein manifold with eigenvalues in the range $0 < k_V^2 < 2$.

5. Quasinormal frequencies of the scalar type electromagnetic field

Here we extend the results of the previous section and calculate exactly the QNF of the scalar type electromagnetic field propagating on the topological black holes (8). In a similar way to the previous section we take

$$\Phi_S = \mathrm{e}^{-i\omega t} R_S(r), \tag{43}$$

and from Eq. (5) we obtain that the function R_S must be a solution of the differential equation

$$f\frac{\mathrm{d}^2 R_S}{\mathrm{d}r^2} + \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}R_S}{\mathrm{d}r} + \left(\frac{\omega^2}{f} + \frac{1}{2r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{3f}{4r^2} - \frac{k_S^2}{r^2}\right) R_S = 0. \tag{44}$$

To solve exactly this equation we use the variable v defined in the formula (15) to find that it transforms into

$$\frac{\mathrm{d}^2 R_S}{\mathrm{d}v^2} + \left(\frac{1}{v} - \frac{1/2}{1-v}\right)\frac{\mathrm{d}R_S}{\mathrm{d}v} + \left(\frac{F + H + 1/16}{v(1-v)} + \frac{F}{v^2} + \frac{1/16}{(1-v)^2}\right) R_S = 0, \tag{45}$$

with F defined in the expression (17) and

$$H = \frac{k_S^2}{4p} + \frac{3}{16}. \tag{46}$$

To solve Eq. (45) we propose that the radial function R_S takes the form

$$R_S = (1-v)^{A_S} v^{B_S} R_{2S}, \tag{47}$$

with the quantities A_S and B_S being solutions of the algebraic equations

$$A_S^2 - \frac{A_S}{2} + \frac{1}{16} = 0, \qquad B_S^2 + F = 0, \tag{48}$$

to find that the function R_{2S} is a solution of the differential equation

$$v(1-v)\frac{\mathrm{d}^2 R_{2S}}{\mathrm{d}v^2} + (2B_S + 1 - (2B_S + 2A_S + 3/2)v)\frac{\mathrm{d}R_{2S}}{\mathrm{d}v} - (2A_S B_S + B_S/2 + A_S - 1/16 - F - H)R_{2S} = 0. \tag{49}$$

This equation is of hypergeometric type (see Eq. (21)) with parameters

$$\alpha_S = A_S + B_S + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{\tfrac{k_S^2+p}{p}},$$
$$\beta_S = A_S + B_S + \tfrac{1}{4} - \tfrac{1}{2}\sqrt{\tfrac{k_S^2+p}{p}},$$
$$\gamma_S = 2B_S + 1. \qquad (50)$$

In what follows we choose $A_S = 1/4$, $B_S = i\omega q/2$, and we notice that the parameters $\alpha_S, \beta_S, \gamma_S$ of the hypergeometric function fulfill $\gamma_S - \beta_S - \alpha_S = 0$. From these results we obtain that the radial function R_S is equal to [34-36]

$$R_S = (1-v)^{1/4} v^{i\omega q/2}\big(K_3\, {}_2F_1(\alpha_S, \beta_S; \gamma_S; v) + K_4 v^{1-\gamma_S}\, {}_2F_1(\alpha_S - \gamma_S + 1, \beta_S - \gamma_S + 1; 2-\gamma_S; v)\big), \qquad (51)$$

where K_3 and K_4 are constants. Near the event horizon ($v = 0$) of the topological black holes (8) this function behaves in a similar way to Eq. (26). To fulfill the boundary condition i) of the QNM we must take $K_3 = 0$ and the function R_S that satisfies this boundary condition is

$$R_S = K_4(1-v)^{1/4} v^{-i\omega q/2}\, {}_2F_1(\alpha_S - \gamma_S + 1, \beta_S - \gamma_S + 1; 2-\gamma_S; v) = K_4(1-v)^{1/4} v^{-i\omega q/2}\, {}_2F_1(\hat{\alpha}_S, \hat{\beta}_S; \hat{\gamma}_S; v), \qquad (52)$$

with

$$\hat{\alpha}_S = \alpha_S - \gamma_S + 1, \qquad \hat{\beta}_S = \beta_S - \gamma_S + 1,$$
$$\hat{\gamma}_S = 2 - \gamma_S. \qquad (53)$$

It is convenient to notice that the new parameters $\hat{\alpha}_S, \hat{\beta}_S, \hat{\gamma}_S$ also fulfill $\hat{\gamma}_S - \hat{\alpha}_S - \hat{\beta}_S = 0$. As for the vector type electromagnetic field, due to this fact we cannot use the Kummer property of the hypergeometric function [34-36] to analyze the behavior of the scalar type electromagnetic field as $r \to +\infty$ ($v \to 1$). Taking into account the property (30) of the hypergeometric function we find that the function R_S satisfying the boundary condition near the horizon takes the form

$$R_S = K_4 v^{-i\omega q/2}(1-v)^{1/4} \frac{(-1)\Gamma(\hat{\gamma}_S)}{\Gamma(\hat{\alpha}_S)\Gamma(\hat{\beta}_S)} \times \sum_{s=0}^{\infty} \frac{(\hat{\alpha}_S)_s (\hat{\beta}_S)_s}{(s!)^2} (1-v)^s [\ln(1-v) - 2\psi(s+1) + \psi(\hat{\alpha}_S + s) + \psi(\hat{\beta}_S + s)] \qquad (54)$$

in the variable $1 - v$. As $r \to +\infty$ ($v \to 1$) this function behaves as

$$R_S \approx (1-v)^{1/4}[\ln(1-v) - 2\psi(1) + \psi(\hat{\alpha}_S) + \psi(\hat{\beta}_S)] \approx \frac{L_1}{r^{1/2}} + \frac{L_2 \ln(r)}{r^{1/2}}, \qquad (55)$$

where L_1 and L_2 are constants. We notice that in the previous formula both terms go to zero as $r \to +\infty$. Thus the function R_S of the expression (52) that fulfills the boundary condition near the horizon, for all the frequencies, it also satisfies the boundary condition ii) of the QNM, thus, we shall obtain a continuous spectrum of QNF for the scalar type electromagnetic field (see for example Refs. 37 and 38). Nevertheless we expect to obtain a discrete set of quasinormal frequencies determined by the physical parameters of the black hole and the field [1,2]. Therefore we must make a careful analysis of the behavior of the radial function R_S as $r \to \infty$ before we impose the boundary condition ii) of the QNM.

We find a similar example in the exact calculation of the QNF for the electromagnetic field propagating on an asymptotically Lifshitz black hole [33]. In the previous reference, to calculate the QNF of the electromagnetic field the boundary condition in the asymptotic region was modified to get a discrete spectrum of QNF. We see that in Ref. 33 it is proposed that for calculating the QNF of the electromagnetic field in the cases where the function R_S is well behaved as $r \to +\infty$ we must impose as a boundary condition that the leading term of the asymptotic behavior must be canceled.

Following Ref. 33 to calculate the QNF of the scalar type electromagnetic field in the topological black holes (8) we propose that instead of the boundary condition ii) of the previous section, in the asymptotic region we must impose as boundary condition that the leading term in the asymptotic behavior (55) of the radial function R_S must be canceled. We notice that for the vector type electromagnetic field when we calculate its QNF in the previous section we also cancel the leading term in the asymptotic behavior of the radial function R_V, but for the vector type electromagnetic field the leading term is divergent as $r \to +\infty$. Thus for the vector type electromagnetic field the new boundary condition as $r \to +\infty$ also can be used to compute its QNF and we obtain the same results of Sec. 4. We think that the new boundary condition that we impose at the asymptotic region is a natural generalization of the boundary condition ii).

In a similar way to Ref. 33 and motivated by the results of the previous section we assume that the parameters $\hat{\alpha}_S$ and $\hat{\beta}_S$ take the values

$$\hat{\alpha}_S = -n, \qquad \hat{\beta}_S = -n, \qquad n = 0, 1, 2, \ldots, \qquad (56)$$

and with these values of the parameters we verify whether the radial function R_S of the formula (52) satisfies the new boundary condition of the QNM as $r \to +\infty$. Thus taking $\hat{\alpha}_S = -n$ we get that this radial function becomes

$$R_S = K_4 v^{-i\omega q/2}(1-v)^{1/4}\, {}_2F_1(-n, \hat{\beta}_S; \hat{\gamma}_S; v). \qquad (57)$$

Using the property (35) of the hypergeometric function we find that as $r \to +\infty$ the radial function (57) behaves as

$$R_S \approx \left(\frac{3p}{\Lambda}\right)^{1/4} \frac{1}{r^{1/2}}, \qquad (58)$$

that is, we cancel the leading term of the asymptotic expansion (55) for R_S and as $r \to +\infty$ the radial function (57) fulfills the new boundary condition of the QNM. Something similar happens when we take $\hat{\beta}_S = -n$. Therefore the QNF of the scalar type electromagnetic field are determined by the conditions (56) and they are equal to

$$\omega_S = \pm\frac{1}{q}\sqrt{\frac{k_S^2 - |p|}{|p|}} - i\frac{2}{q}\left(n + \frac{1}{2}\right). \qquad (59)$$

In contrast to the QNF of the vector type electromagnetic field, the previous QNF do not depend on the parameter K. Recalling that for $K = 0, \pm 1$ the eigenvalues k_S^2 are non negative [32], for the three values of K and for $k_S^2 > |p|$ we have complex QNF that are stable since $\mathbb{Im}(\omega_S) < 0$. For $k_S^2 < |p|$ the QNF (59) transform into

$$\omega_S = \pm\frac{i}{q}\sqrt{\frac{|p| - k_S^2}{|p|}} - i\frac{2}{q}\left(n + \frac{1}{2}\right), \qquad (60)$$

that are purely imaginary. Thus depending on the values of $|p|$ and k_S^2 we get complex or purely imaginary QNF for the scalar type electromagnetic field. For the minus sign of the square root in the formula (60) we have stable QNM, but for the plus sign of the square root we obtain QNF with $\mathbb{Im}(\omega_S) > 0$ when we fulfill the following inequality

$$|p| - k_S^2 > 4|p|\left(n + \frac{1}{2}\right)^2. \qquad (61)$$

In a straightforward way we verify that this inequality cannot be satisfied for the allowed values of the physical parameters, that is, for $k_S^2 < |p|$ also we obtain stable QNM. Thus for the scalar type electromagnetic field we find only stable QNM in the five dimensional topological black holes (8).

6. Discussion

In the previous sections we calculate exactly the QNF of the vector type and the scalar type electromagnetic fields propagating on the topological black holes (8). It is convenient to notice that for the three values of the parameter K we state the radial problem in a common form and we solve simultaneously the differential equations. It is convenient to notice that the method previously used to calculate the QNF (37) and (59) is slightly different from the used in other references, since the special values of the parameters for the hypergeometric functions that we obtain in the topological black hole (8), force us to impose the conditions (33) and (56) and then verify that for these values of the parameters the radial functions satisfy the boundary condition at infinity. Usually it possible to employ the Kummer property of the hypergeometric function and to choose the appropriate behavior at the boundaries by imposing the analogue of the conditions (33) and (56) [18-25].

Depending on the physical parameters for the scalar type and vector type electromagnetic fields we obtain complex QNF or purely imaginary QNF. We find that the QNM of the electromagnetic field are stable, except for the topological black holes with $K = -1$ of the plus branch for which we find that for small values of the parameter $|p|$ and of the eigenvalues k_V^2 for the vector harmonics, the QNM of the vector type electromagnetic field would be unstable if the three-dimensional base manifold has eigenvalues k_V^2 satisfying $0 < k_V^2 < 2$. Therefore, as noted previously for other backgrounds in Refs. 32 and 39, the eigenvalues of the scalar and vector harmonics of the base manifold play a relevant role in the analysis of the classical stability of the black holes under perturbations.

Comparing the expressions (37) and (59) for the QNF of the vector type and scalar type electromagnetic fields, we see that they are not isospectral since the terms in square roots show a different dependence on the parameter p. Also, notice that the QNF of the vector type electromagnetic field depends on the scalar curvature K, whereas the QNF of the scalar type electromagnetic field are independent of K. Furthermore the QNF of the vector type field depend on the overtone number n in the form $(n + 1)$, whereas the QNF of the scalar type field depend on n in the form $(n + 1/2)$.

Finally considering that the Hawking temperature of the topological black holes (8) is [28]

$$T_H = \frac{1}{2\pi q}, \qquad (62)$$

for the vector type electromagnetic field we can write its QNF (37) as

$$\omega_V = \pm 2\pi T_H\sqrt{\frac{k_V^2 + 2K}{|p|}} - i4\pi T_H(n + 1), \qquad (63)$$

and for the scalar type electromagnetic field its QNF (59) as

$$\omega_S = \pm 2\pi T_H\sqrt{\frac{k_S^2 - |p|}{|p|}} - i4\pi T_H\left(n + \frac{1}{2}\right). \qquad (64)$$

Thus for the five-dimensional topological black holes (8) the QNF are proportional to its Hawking temperature.

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i. Notice that the quantity v varies over the range $0 < v < 1$.

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