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The universal optimal relations of the allocation and effectiveness of the heat exchangers for power plants with n Carnot-like cycles

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A model of irreversible Carnot-like power plant with n Carnot-like cycles is optimized. The irreversibilities of each cycle are: finite rate heat transfer between the working fluid and the external heat sources, internal dissipation of the working fluid, and heat leak between reservoirs; is extended to two or more of this combined model. Applying the Bellman' Principle, we find the optimal recurrence relations for the allocation of the heat exchangers for thes power plants. The optimal allocation or effectiveness is determined by two design rules, applied alternatively: internal thermal conductance fixed or areas fixed. The optimal recurrence relations obtained for this combined model are invariant to the power and efficiency and to the heat transfer law.

Keywords: Irreversibilities; Carnot; allocation; conductance; effectiveness; area; optimal.

Se optimiza un modelo de planta de potencia irreversible con n ciclos tipo Carnot. Las irreversibilidades de cada uno de los ciclos son: transferencia finita de calor entre el fluido de trabajo y los depósitos de temperatura, disipación del fluido de trabajo y fuga de calor entre ambos depósitos; es extendido a dos o más de este ciclo combinado. Aplicando el Principio de Bellman, encontramos las relaciones de recurrencia óptimas para la dimensión y efectividad de los intercambiadores de calor para modelo combinado. La óptima dimensión y efectividad se determina mediante dos reglas de diseño, aplicadas alternativamente: conductancias térmicas restringidas ó área total constante. Las relaciones óptimas para la dimensión ó efectividad obtenidas para este modelo combinado son invariantes a la potencia y eficiencia y a la ley de transferencia de calor.

Descriptores: Irreversibilidades; Carnot; dimensión; efectividad; conductancia; área; óptima.

PACS: 01.40G; 05.70.-a; 64.70.F

1. Introduction

Recently in [1], a methodology of optimization was applied to an irreversible Carnot-like power plant, where the characteristic parameters were: the allocation or effectiveness of the heat exchangers of this plant. Although this methodology is applicable to any Carnot-like power plant, a standard irreversible Carnot-like cycle was chosen because of its simplicity to account for the main irreversibilities that usually arise in real heat engines [2]: "finite rate heat transfer between the working fluid and the external heat sources, internal dissipation of the working fluid, and heat leak rate between reservoirs". The above standard irreversible Carnot-like cycle has been studied at length for many objective functions, different transfer heat laws and several characteristic parameters (see [2-24] for more details).

The maximum power and efficiency have been obtained in [3-6]. In general, these optimizations were performed with respect to only one characteristic parameter: internal isentropic temperature ratio (x) including the time too [18]. However, Bejan and Lewins [15,20] have considered the optimization with respect to other parameters (see also the reviews of [17-18] for more details): "the allocation, cost and effectiveness of the heat exchangers of the hot and cold sides". On the other hand, effects of heat transfer laws or when a property is independent of the heat transfer law for this Carnot-like cyclic model, have been discussed in several works, [18-23], and so on.

On the other hand, n-stage combined Carnot cycle has been presented, by the law of heat conduction, in [24-26] optimizing the specific power and efficiency for the characteristic parameters: isentropic temperature ratios , and effectiveness (other combined cycles can be found in [27-36], coupled heat devices in [37-38]; see also the references therein included).

In this paper we extend the results of allocation or effectiveness for one cycle to two or more combined cycles of the same irreversible Carnot-like power plant using iteratively the Bellman' Principle [39] which has been successfully applied in Refs. 13 and 14. We found optimal recurrence relations remarkable for the irreversible power plant with n Carnot-like cycles for two constraints (design rules): constrained internal thermal conductance or fixed total area of the heat exchangers from hot and cold side.

This paper is organized as follows. In Sec. 2 we present the irreversible power plant model and the functional relation between power and efficiency which extends to the corollary presented in [1] to the irreversible power plant. Section 3 presents the optimization of the allocation and effectiveness, by two design rules, of heat exchangers for the Carnot-like n-cycles of the power plant. Section 4 is devoted Conclusions.

2. Power plant with n Carnot-like cycles

The power plant with n Carnot-like cycles is shown in Fig. 1. Each cycle satisfies the conditions expressed in [2,4]: leak heat rate Q and finite heat transfer rates Q_i and internal dissipations of the working fluid expressed by constants $I_i(i{=}1\ldots n)$ such that $I_i = (\Delta S_{i+1}/\Delta S_i) \geq 1; i = 1\ldots n$ [4] which make the Claussius inequality to become equality (for more details see [1,24]). Each cycle of the power plant consists of two isothermal and two adiabatic processes. Denoting, for each cycle, the temperatures of the working fluid in the hot and cold isothermal processes as T_{2i-1} and $T_{2i}(i = 1\ldots n)$, respectively, and the end temperatures as T_H and T_L.

Following to [1,25], the thermal efficiency of the power plant is given by (this functional form for only a cycle has appeared in Refs. 1, 2, and 17):

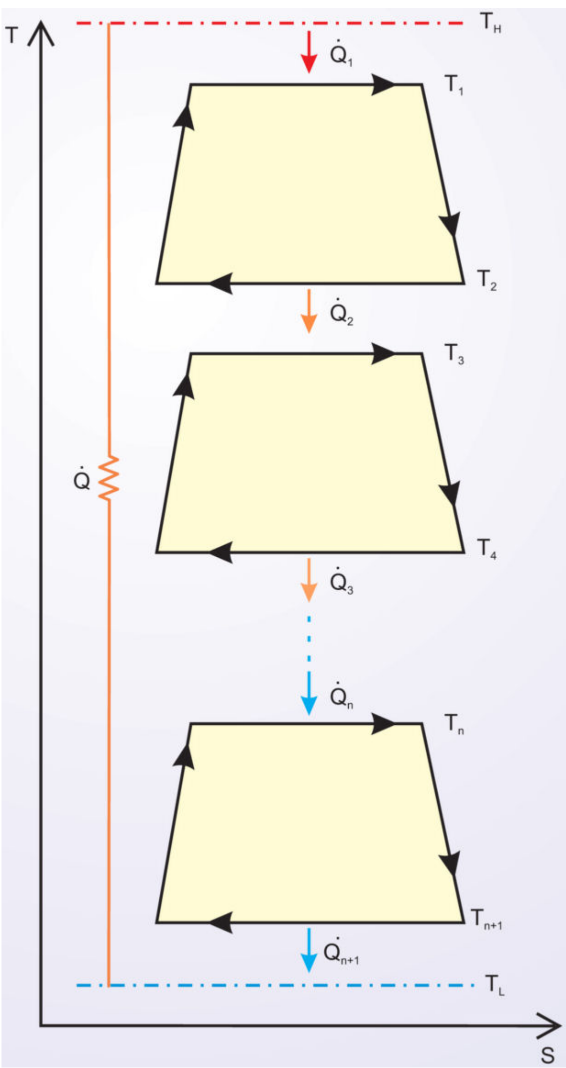

FIGURE 1. A power plant with n Carnot-like cycles, with heat leak rate, finite heat transfer rates, and internal dissipations of the working fluid in each cycle.

$$\eta = \frac{P}{f(x_1, \ldots.., x_n)P + Q} \tag{1}$$

where $x_i = T_{2i}/T_{2i-1}$ corresponds to the isentropic temperature ratio for each cycle $i(i = 1, \ldots., n)$, P is the power of the plant, Q is the leak heat rate and the function

$$f(x_1, \ldots.., x_n) = \frac{1}{1 - (\prod_{i=1}^{n} I_i x_i)} \tag{2}$$

The Eq. (1) is obtained of the following way: according to the second Law of the Thermodynamics, for each $i(= 1, 2, \ldots, n)$, we have (cfr. with [25]),

$$Q_{i+1} = I_i x_i Q_i \tag{3}$$

Thus,

$$\begin{aligned} Q_{n+1} &= I_n x_n Q_n = \ldots. = I_i x_i Q_i \\ &= \ldots\ldots = I_1 x_1 Q_1 \end{aligned} \tag{4}$$

Now, using the Eqs. (3) and (4), the efficiency is given by,

$$\begin{aligned} \eta &= \frac{P}{Q_H} = \frac{Q_H - Q_L}{Q_H} = \frac{Q_1 - Q_{n+1}}{Q_H} \\ &= \left(1 - \frac{Q_{n+1}}{Q_1}\right)\frac{Q_1}{Q_1} = \left(1 - \left(\prod_{i=1}^{n} I_i x_i\right)\right)\frac{Q_1}{Q_H} \end{aligned} \tag{5}$$

From the Eq. (5), Q_1 is equal to:

$$Q_1 = \frac{P}{1 - (\prod_{i=1}^{n} I_i x_i)} \tag{6}$$

And from the Eq. (6), (1) is obtained.

The Eq. (1) extends to the Eq. (6) of [1]. Therefore, we can seek, as in [1], invariant optimal relations, by following the spirit of Carnot's work, for other characteristic parameters, which can be independent from the heat transfer law, of the irreversible power plant.

Also, the Eq. (1) say us that it is enough analyzes only for power and efficiency (for other operation regimes, *e.g.* algebraic combination of power and/or efficiency that have thermodynamic meaning and satisfy imposed power conditions: see Eq. (7) below and for more details see [1]). For instance, the functional expression of the efficiency, power and the heat transfer, without require of algebraic explicit expressions, have been used in [2], for two constraints (design rules): constrained internal thermal conductance or fixed total area of the heat exchangers from hot and cold side. The isentropic temperature ratio do not has this condition [1,2,3,13,24-26].

Now, if we take $z \neq x_i(i = 1, 2\ldots.n)$ as other independent variable the Corollary given at [1] can be paraphrased as:

"The power P achieves a maximum value in z_{mp} if and only if the efficiency η achieves a maximum value in the value: $z_{mp} = z_{me}$"

Thus, in the optimization of power plant, with respect to z it is enough to find the maximum power by,

$$\left.\frac{\partial P}{\partial z}\right|_{z_{mp}=z_{me}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 P}{\partial z^2}\right|_{z_{mp}=z_{me}} < 0 \qquad (7)$$

The optimization performed, with respect to will be: a property independent from the heat transfer law. This a remarkable conclusion of the Eq. (1) is that it can find the maximum for one and only one the power (see the criterion of [2]). For example the power, for power plant model, by

$$\left.\frac{\partial P}{\partial z}\right|_{z_{mp}=z_{me}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 P}{\partial z^2}\right|_{z_{mp}=z_{me}} < 0 \qquad (8)$$

We show the above as follows: Let $z \neq x_i (i = 1, 2....n)$ and the vector $x = (x_1, ..., x_n)$ and z_{mp} the point in which the power achieves a maximum value, then,

$$\left.\frac{\partial P}{\partial z}\right|_{z_{mp}=z_{mp}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 P}{\partial z^2}\right|_{z_{mp}} < 0.$$

The power and the efficiency satisfy the functional relationship given by the Eq. (1). Deriving (1) with respect to:

$$\frac{\partial \eta}{\partial z} = \frac{Q(\frac{\partial P}{\partial z})}{[f(x)P + Q]^2} \qquad (9)$$

since, we can suppose that Q does not depend of the variable z.

Therefore,

$$\left.\frac{\partial \eta}{\partial z}\right|_{z_{me}} = 0 = \left.\frac{\partial P}{\partial z}\right|_{z_{mp}};$$

where z_{me} is the point in which the efficiency η achieves a maximum value which implies that their roots are the same $z_{mp} = z_{me}$. Also, it is easily seen that for $z_{mp} = z_{me}$, the efficiency reach a maximum since

$$\left.\frac{\partial^2 \eta}{\partial z^2}\right|_{z_{me}=z_{me}} = \frac{Q\left(\left.\frac{\partial^2 P}{\partial z^2}\right|_{z_{me}=z_{me}}\right)}{[f(x)P + Q]^2} < 0 \qquad (10)$$

In all the above calculations none transfer heat law has been used.

Therefore, it is enough to have one, and only one, algebraic expression of the power and *any heat transfer law*. We choose for our convenience the power and the conduction heat transfer Law since they are algebraically the simplest. In the optimization of the power of the plant for two constraints (design rules): constrained internal thermal conductance ($z = \phi$) or fixed total area ($z = \psi$) of the heat exchangers from hot and cold side of the heat exchangers from hot and cold side.

3. Universal optimal relations for the allocation and effectiveness of the heat exchangers of the power plant

The optimization of power and efficiency with respect to is well known for combined cycles [13,24-26]. Henceforth, **x** will be fixed and we will assume that the law of heat transfer rate can be any law, included also the heat leak rate. Essentially, we will treat the following two design rules: internal thermal conductance fixed; or areas fixed for heat exchangers; which will be applied alternatively.

The first design rule is that the internal conductance of the Carnot-like cycle is constrained to: $\sum_{i=1}^{n} \alpha_i = \Gamma$; where Γ is a constant, which is applied to the allocation of the heat exchangers from hot and cold side with the same overall heat transfer coefficient U by unit of area A in both ends of the cycle i and $\alpha_i, \alpha_{i+1} (I = 1, 2....n - 1)$ are the thermal conductance correspondent to the finite heat transfers of the hot/cold sides for this cycle. Thus,

$$\sum_{i=1}^{n} UA_i = \Gamma \qquad (11)$$

where A_i, A_{i+1} are heat transfer areas on hot/cold sides of the cycle $i(i = 1, 2.....n - 1)$.

Alternatively, the second design rule is that the total area is constrained by: $\sum_{i=1}^{n} A_i = A$; where A_i, A_{i+1} are heat transfer areas on cold and hot side for the cycle i. Now, the total area (A) is fixed, but when distributed it has distinct overall heat transfer coefficients and hence different effectiveness on each one of the hot and cold sides. As $\alpha_i = U_i A_i$ (see [1 and 12]), then,

$$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{\alpha_i}{U_i} \qquad (12)$$

where U_i, U_{i+1} are the overall heat transfer coefficients on cold/hot sides of the cycle $i(i = 1, 2.....n)$. From the result of the Sec. 2, it is enough to found "*The maximum power of each Carnot-like cycle* $i(i = 1, 2.....n)$ *for the two design rules (11) and (12)*"

Indeed, the dimensionless power output (Eq. (2.5) for one cycle in [3]) for each cycle i is given by:

$$p_i = \frac{p_i}{\alpha_i T_{2i-1}} = \frac{h(x_i)}{\frac{1}{\alpha_1} + \frac{I_i}{\alpha_{1+1}}} \qquad (13)$$

here $h(x_i)$ is a function which depends on the heat transfer law since $x_i = T_{2i}/T_{2i-1}$ is the isentropic temperature ratio of the cycle i ([1,2,3,13,24-26]); and α_i, α_{i+1} $i(i = 1, 2.....n - 1)$ are the thermal conductance correspondent to the finite heat transfers of the cold/hot sides for this same cycle.

Next, applying the Bellman' Principle [39], in form iterative, we can obtain the optimal relations for the allocation of the heat exchangers of the power plant (Fig. 1). This principle will be applied as it is indicated in [20]: "*to state that every part of an optimum path is optimal*".

3.1. Constrained internal thermal conductance

The n thermal conductance can be written as: $\alpha_i = UA_1$; $i = 1 \ldots n+1$ where U is overall heat transfer coefficient and A_i; $i = 1 \ldots n+1$ are the available areas for heat transfer. Thus, for the first optimization we can take for this first design rule:

$$\alpha_1 + \alpha_2 = \Gamma_1$$

where $\Gamma_1 = \Gamma - \sum_{i=3}^{n} \alpha_i$ is supposed to be a constant. Equivalently,

$$\frac{\alpha_1 + \alpha_2}{\Gamma_1} = 1$$

Fixing the temperature $T_j (j = 3 \ldots n+1)$ and applying only for the first cycle:

$$\frac{a_1\alpha_1 + \alpha_2}{\Gamma_1} = 1; \quad a_1 = 1$$

In parameterizing,

$$\phi = \frac{\alpha_1}{\Gamma_1}; 1 - a_1\phi = \frac{\alpha_2}{\Gamma_1}$$

According to the Eq. (13), in optimizing

$$\left(\frac{1}{\frac{a_1}{\phi} + \frac{I_1}{1-a_1\phi}} \right)$$

with respect to ϕ; we obtain:

$$\phi_1 = \frac{1}{1+\sqrt{I_1}}$$

since $a_1 = 1$. Solving,

$$\frac{1-a_1\phi}{\phi_1} = \sqrt{I_1} = \frac{\alpha_2}{\alpha_1};$$
$$\alpha_2 = \sqrt{I_1}\alpha_1.$$

Continuing of this way we arrive to the following optimal recurrence relation:

$$\alpha_n \sqrt{I_{n-1}}\alpha_1, \quad n = 2, 3 \ldots\ldots \tag{14}$$

and

$$a_n = \sum_{j=0}^{n-1} \sqrt{I_j}; I_0 = 1.$$

From the Eqs. (11) and (14) we have: the areas de transference of heat decreases from T_H to T_L:

$$A_1 \geq A_2 \geq \ldots\ldots\ldots A_{n-1} \geq A_n \tag{15}$$

and the equality is fulfilled if there is not internal dissipation in the n cycles.

3.2. Constrained areas of heat exchangers

For simplicity, we suppose $I_i = 1 (i = 1 \ldots. n)$, without internal irreversibilities. Applying the design second design rule (Eq. (12)),

$$A_1 + A_2 = A_1$$

where $A_1 = A - \sum_{i=3}^{n} A_i$ is supposed a constant and fixing the temperature $T_j (j = 3 \ldots\ldots n+1)$. How,

$$A_1 = \frac{\alpha_1}{U_1}; \quad A_2 = \frac{\alpha_2}{U_2}$$

then

$$A_1 = a_1\alpha_1 + u_1\alpha_2$$

where $u_1 = U_1/U_2$; $a_1 = 1$. In parameterizing,

$$\psi = \frac{\alpha_1}{A_1}; 1 - a_1\psi = \frac{\alpha_2}{A_1}$$

From according to the Eq. (13):

$$\frac{1}{\frac{a_1}{\psi} + \frac{1}{(1-a_1\psi)u_1}},$$

the first optimization with respect to ψ, gives:

$$\psi_1 = \frac{\sqrt{u_1}}{1 + a_1\sqrt{u_1}}$$

Then,

$$\frac{1 - a_1\psi_1}{\psi_1} = \frac{1 + \sqrt{u_1}(a_1 - 1)}{\sqrt{u_1}}.$$

Solving,

$$A_2 = \frac{A_1}{\sqrt{a_1 u_1}}$$

and, the results of [1] are recovered.

Continuing of this way arrive us to the following optimal recurrence relation:

$$A_n = \frac{1 + \sqrt{u_{n-1}}(a_{n-1} - 1)}{1 + a_{n-1}\sqrt{u_{n-1}}} A_1 \tag{16}$$

with

$$u_{n-1} = U_1/U_n; \quad a_{n-1} = a_2 + a_{n-2},$$
$$n > 2; \quad a_2 = 1 + \frac{1}{\sqrt{a_1 u_1}}; \quad a_1 = 1.$$

3.3. Some special cases

To illustrate the optimal relations (14 and 16) we can choose a power plant with 2 or 3 like-like-Carnot cycles:

a). Case constrained internal thermal conductance

In this case, from the Eq. (14), we obtain the conductance: $\alpha_2 = \sqrt{I_1}\alpha_1$; $\alpha_3 = \sqrt{I_2}\alpha_1$ and $\alpha_4 = \sqrt{I_3}\alpha_1$ for only three cycles. If $I_j = 1$; $j = 1, 2, 3$ (without internal irreversibilities). Then, the optimal conductance without internal irreversibilities are obtained: $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_1 = (1/4)\Gamma_4$. For two cycles: $\alpha_3 = \alpha_2 = \alpha_1 = (1/3)\Gamma_3$ only and so on. This last result was reported in [13] using only the heat conduction law. Thus, the expressions obtained from (14) are extended to any heat transfer law.

b). Constrained areas:

In this case, from the Eq. (16), we obtain the areas, for two cycles,

$$A_1 = \frac{\sqrt{u_1}}{1+\sqrt{u_1}}A; \quad A_2 = \frac{A}{1+\sqrt{u_1}};$$
$$A_3 = \frac{u_1 + \sqrt{u_1}\sqrt{u_2}}{(\sqrt{u_1} + \sqrt{u_2 u_1} + \sqrt{u_2})(1+\sqrt{u_1})}.$$

And for three cycles,

$$A_2 = \frac{1}{\sqrt{u_1}}A_1; \quad A_3 = \frac{\sqrt{u_1}+\sqrt{u_2}}{\sqrt{u_1}+\sqrt{u_2 u_1}+\sqrt{u_2}}A_1$$
$$A_4 = \frac{\sqrt{u_1}+\sqrt{u_2}+\sqrt{u_2 u_1}+\sqrt{u_3 u_1}+\sqrt{u_3 u_2}}{\sqrt{u_1}+\sqrt{u_2}+\sqrt{u_1}+2\sqrt{u_3 u_1}+2\sqrt{u_3 u_2}+\sqrt{u_3 u_2 u_1}}A_1$$

where $u_{j-1} = U_1/U_j$; $I_j = 1$; $j = 2, 3, 4$; and so on. The expressions obtained by (16) are extended for any heat transfer law.

4. Conclusions

We have found and determined the optimal allocation of heat exchangers of a power plant with n Carnot-like cycles. We must note that the optimal recurrence relations (Eqs. (14 and 16)) found for this model are valid for any heat transfer law and have not been reported in the literature previously; specially the Eq. (16). Thus, following the spirit of Carnot's work, we have found invariant optimal relations for power and efficiency maximum which are valid for any heat transfer law. Moreover, these relations can be satisfied for other operation regimes, *e.g.* algebraic combination of power and/or efficiency that have thermodynamic meaning and satisfy the conditions imposed to the power (Eq. (7)) which can be carried out as in [1]. Nevertheless, the optimal isentropic temperature ratios depend on the heat transfer law and the operation regime of the engine as is discussed in [1,2,3,13,24-26].

Finally, the above equations can be extended to n Carnot-like cycles presented here including other heat transfer rate laws by the substitution of either of (14) or (16) in the objective function (algebraic combination of power and/or efficiency) and optimize only for the isentropic temperature ratio. However, the latter requires a comprehensive study of the implications for the power plant considered. We will study such implications as future work.

Appendix

For the second optimization of the first design rule, the constraint is now:

$$\frac{a_2\alpha_1 + \alpha_3}{\Gamma_2} = 1$$

with $a_2 = 1 + \sqrt{I_1}$; $\Gamma_2 = \Gamma - \sum_{i=4}^{n}\alpha_i$ a constant and fixing the temperature $T_j (j = 4....n+1)$ and applying only for the second cycle. In parameterizing,

$$\phi = \frac{\alpha_1}{\Gamma_2}; \quad 1 - a_2\phi = \frac{\alpha_3}{\Gamma_2}$$

Similarly, in optimizing

$$\left(\frac{1}{\frac{a_2}{\phi} + \frac{I_2}{1-a_2\phi}}\right)$$

with respect to ϕ; we obtain:

$$\phi_2 = \frac{1}{a_2 + \sqrt{I_2}}$$

Solving

$$\frac{1 - a_2\phi_2}{\phi_2} = \sqrt{I_2} = \frac{\alpha_3}{\alpha_1}$$

so,

$$\alpha_3 = \sqrt{I_2}\alpha_1$$

and $a_3 = 1 + \sqrt{I_1} + \sqrt{I_2}$. Continuing of this form arrive us to the Eq. (14).

We can show the relation (14) by Mathematical Induction [38]. Indeed for $n = 2$ is true. We suppose that for $n = k$ is true and we must show for $n = k + 1$. From the hypothesis of induction the following is fulfilled. For the $k - th$ optimization, the constraint is now,

$$\frac{a_k\alpha_1 + \alpha_{k+1}}{\Gamma_k} = 1$$

where $a_k = \sum_{j=0}^{k-1}\sqrt{I_j}$. In parameterizing

$$\phi = \frac{\alpha_1}{\Gamma_k}; 1 - a_k\phi = \frac{\alpha_{k+1}}{\Gamma_k}$$

In optimizing

$$\left(\frac{1}{\frac{a_k}{\phi}+\frac{I_k}{1-a_k\phi}}\right),$$

we obtain:

$$\phi_k=\frac{1}{a_k+\sqrt{I_k}}.$$

Solving,

$$\frac{1-a_k\phi_k}{\phi_k}=\sqrt{I_k}=\frac{\alpha_{k+1}}{\alpha_1}.$$

Therefore,

$$\alpha_{k+1}=\sqrt{I_k}\alpha_1$$

and

$$a_{k+1}=\sum_{j=0}^{k}\sqrt{I_j}$$

And the proof is complete.

For the second optimization of the second design rule, the constraint is now,

$$a_2A_1+\frac{A_3}{u_2}=A_2$$

where

$$a_2=\frac{1+\sqrt{a_1u_1}}{\sqrt{a_1u_1}};\quad u_2=\frac{U_1}{U_3};$$

$$A_2=A-\sum_{i=4}^{n}A_i$$

is a constant and fixing the temperature $T_j(j=4....n+1)$. In parameterizing,

$$\psi=\frac{A_1}{A};1-a_2\psi=\frac{A_3}{A}$$

where

$$a_2=1+\frac{1}{\sqrt{a_1U_1}}.$$

From (13)

$$\frac{1}{\frac{a_2}{\psi}+\frac{1}{(1-a_2\psi)u_2}},$$

the optimization with respect to ψ, gives,

$$\psi_2=\frac{\sqrt{u_2}}{1+a_2\sqrt{u_2}}$$

and

$$\frac{1-a_2\psi_2}{\psi_2}=\frac{1+\sqrt{u_2}(a_2-1)}{1+a_2\sqrt{u_2}}$$

Thus,

$$A_3=\frac{1+\sqrt{u_2}(a_2-1)}{\sqrt{u_2}}A_1$$

Continuing of this form arrive us to the Eq. (15). In analogous form, the proof of the relations (15) can be carry out by Mathematical Induction.

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